Regular Singular Stratified Bundles in Positive Characteristic

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Contents

Introduction

Let k be an algebraically closed field of characteristic $p > 0$. To begin this introduction, we describe the main objects of study of this dissertation: *regular singular stratified bundles* on smooth k-varieties X.

The notion of a stratification goes back at least to Grothendieck ([Gro68]), and it has close ties to his formulation of descent theory: A stratification on an object E of a category fibered over the category of X-schemes is an “infinitesimal descent datum relative to k” on E. If E is an $\mathcal{O}_X$-module, then since X is smooth, giving a stratification on E is equivalent to giving E the structure of a left-$\mathscr{D}_{X/k}$-module, where $\mathscr{D}_{X/k}$ is the sheaf of differential operators as developed by Grothendieck in [EGA4, §16]. Of course, in its essence, a $\mathscr{D}_{X/k}$-module is “just” a conceptual way to describe a system of differential equations on X.

The sheaf of $\mathcal{O}_X$-algebras $\mathscr{D}_{X/k}$ is filtered by the degree of differential operators, and if k is a field of characteristic 0, then $\mathscr{D}_{X/k}$ is generated over $\mathcal{O}_X$ by the operators of degree 1, i.e. by derivations of $\mathcal{O}_X$ into itself. This implies that giving a stratification on an $\mathcal{O}_X$-module E is equivalent to giving a flat connection on E. Flat connections over smooth k-varieties have been extensively studied in various contexts; for example in P. Deligne's version of the so called Riemann-Hilbert correspondence in [Del70]: If X is a projective, smooth complex variety, then the category of vector bundles on X with flat connection is (tensor-) equivalent to the category of finite dimensional complex representations of the topological fundamental group of the analytic variety $X(\mathbb{C})$. Deligne's additional insight was that the projectivity condition can be dropped, once one restricts to vector bundles with a “regular singular” flat connection. In his formulation, if $\overline{X}$ is a projective variety compactifying X, a flat connection is regular singular, if the growth of its solutions approaching infinity (i.e. approaching $\overline{X} \setminus X$) is “moderate”. If the compactification $\overline{X}$ is smooth, and if the boundary divisor $D := \overline{X} \setminus X$ has strict normal crossings, then this growth condition can be made precise by requiring that the vector bundle with flat connection extends to a torsion free $\mathcal{O}_{\overline{X}}$-module with flat *logarithmic* connection.

In positive characteristic, flat connections turn out to be not very well-behaved: For example, if $\mathcal{O}_X$ is endowed with the trivial connection, then in the category of vector bundles with flat connection one has $\mathrm{End}(\mathcal{O}_X) = \mathcal{O}_{X^{(1)}} \neq k$, where $X^{(1)}$ is the first frobenius twist of X. On the other hand, the category of vector bundles with stratification does have many of the good properties of the category of flat connections in characteristic 0. In particular, it is a k-linear tannakian category, so we have at our disposal the tools of representation theory, and monodromy groups of stratified vector bundles can be discussed in this framework.

The notion of a regular singular stratified bundle in positive characteristic has been defined by N. Katz and D. Gieseker in [Gie75] in the presence of resolution of singularities: If $\overline{X}$ is a smooth compactification of X, such that the boundary divisor D has strict normal crossings, then just as in the classical situation, one defines a stratified bundle to be regular singular if it extends to a suitable module over the ring $\mathscr{D}_{\overline{X}/k}(\log D)$ of logarithmic differential operators. The main goal of this dissertation is to systematically develop a theory of regular singular stratified bundles without relying on resolution of singularities, and to study the category of these objects. Moreover, part of this dissertation is concerned with the comparison of Gieseker's definition of regular singularity with alternative definitions that are known to be equivalent in characteristic 0.

A major phenomenon appearing only in this characteristic $p > 0$ context is the dichotomy between tame and wild ramification. One of the main results of this text is that a stratified bundle is regular singular with finite monodromy group if and only if it is trivialized on a tamely ramified covering. This is true without assuming the validity of resolution of singularities. As a corollary one obtains that a stratified vector bundle with finite monodromy group is regular singular if and only if it is regular singular on all curves lying in X. This uses results from M. Kerz, A. Schmidt and G. Wiesend, [KS10].

Finally, we come back to P. Deligne's Riemann-Hilbert correspondence: Together with a theorem of Malcev-Grothendieck, it implies that a smooth $\mathbb{C}$-variety is étale simply connected (i.e. $\pi_1^{\text{ét}}(X) = 0$) if and only if on X there are no vector bundles with nontrivial regular singular flat connection. It is an interesting question whether an analogous statement for regular singular stratified bundles in positive characteristic is true. This was conjectured by Gieseker in [Gie75] for projective X, in which case every stratified bundle is regular singular. Gieseker's conjecture is now a theorem of H. Esnault and V. Mehta ([EM10]). For nonproper X, in light of the results of this thesis which relate regular singular bundles to tamely ramified coverings, it is most natural to formulate analogous statements with the tame étale fundamental group $\pi_1^{\text{tame}}(X)$ instead of $\pi_1^{\text{ét}}(X)$. We can prove partial results: We show that $\pi_1^{\text{tame}}(X) = 0$ if every regular singular stratified bundle is trivial, and conversely that if $\pi_1^{\text{tame}}(X) = 0$, then there are no nontrivial regular singular stratified bundles of rank 1. These results do not rely on resolution of singularities.

Leitfaden

In Chapter 1 we study the formal local analog of regular singular stratified bundles on fields of the form $k(\!(x_1, \dots, x_n)\!)$. The guiding picture the reader should keep in mind is the following: Let X and $\overline{X}$ be smooth, separated, finite type schemes over an algebraically closed field of characteristic $p > 0$, such that $X \subseteq \overline{X}$ is an open subscheme, and such that $\overline{X} \setminus X$ is a strict normal crossings divisor. We write $(X, \overline{X})$ for such a situation and call it *good partial compactification* of X. If $x \in \overline{X} \setminus X$ is a closed point, then after choosing coordinates $x_1, \dots x_n$ around x, there is a canonical isomorphism $\operatorname{Frac} \widehat{\mathcal{O}_{\overline{X},x}} = k(\!(x_1, \dots, x_n)\!)$, and there is a canonical morphism

$$\operatorname{Spec} k(\!(x_1, \dots, x_n)\!) \to X.$$

If E is a stratified bundle on X, we can pull back E along this morphism to get a finite dimensional $k(\!(x_1, \dots, x_n)\!)$-vector space with an action of a ring of differential operators. These are the objects studied in Chapter 1. For $n = 1$, they are called iterative differential equations in [MvdP03].

Special emphasis is put on the tannakian point of view, and we generalize parts of [Del90, §9] and [MvdP03] to our context. In particular we show how Picard-Vessiot extensions arise from tannakian theory, and that the tannakian fundamental group of a stratified bundle E on $k(\!(x_1, \ldots, x_n)\!)$ is precisely the group of differential automorphisms of the Picard-Vessiot ring of E. For $n = 1$ this is mostly done in [MvdP03], and many of the arguments from *loc. cit.* carry over.

In Chapter 1 we also give an exposition of Gieseker's result that a stratified bundle E on $k(\!(x_1, \ldots, x_n)\!)$ is regular singular if and only if it is a direct sum of stratified bundles of rank 1. Using the description of the monodromy group of E via Picard-Vessiot extensions, we prove that E is regular singular if and only if its monodromy group is diagonalizable, and we compute various examples.

In Chapter 2 we go back to a global point of view. The sheaves of logarithmic principal parts are carefully defined for finite type morphisms of fine log-schemes, and we analyze their structure in case the morphism is of the form $(\overline{X}, M_D) \to \operatorname{Spec} k$, where M_D is the log-structure associated with a strict normal crossings divisor D on a smooth k-variety $\overline{X}$, and $\operatorname{Spec} k$ carries its trivial log-structure. From this analysis it follows that in this situation the sheaves of principal parts give rise to a "formal groupoid" in the sense of P. Berthelot, [Ber74, II.1]. Thus a large part of the basic theory of logarithmic connections, logarithmic n-connections and logarithmic stratifications follows formally.

Next we define exponents of $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-modules as elements of $\mathbb{Z}_p$ and show that they can be computed formally locally as in Chapter 1.

In Chapter 3 we finally define the notion of a regular singular stratified bundle. This is done in two steps: First one fixes a good partial compactification $(X, \overline{X})$ and defines what it means for a stratified bundle E on X to be $(X, \overline{X})$-regular singular: E is $(X, \overline{X})$-regular singular if it extends to a torsion free $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$-module. It turns out that this notion is fairly well behaved: We prove that the category of $(X, \overline{X})$-regular singular bundles is a full tannakian subcategory of $\operatorname{Strat}(X)$, and that it is invariant under pullback along universal homeomorphisms. Since the extension of E to a $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$-module is not unique, the exponents no longer lie in $\mathbb{Z}_p$, but rather in $\mathbb{Z}_p/\mathbb{Z}$. We also show that once one fixes some additional data, an extension of E to a $\mathscr{D}_{\overline{X}/k}(\log D)$-module *is* unique: If $\tau : \mathbb{Z}_p/\mathbb{Z} \to \mathbb{Z}_p$ is a section of the canonical projection, then there exists a unique extension of E with exponents in the image of τ, and we show that this extension is in fact $\mathcal{O}_{\overline{X}}$-locally free. This is analogous to a construction of P. Deligne in characteristic 0.

Once one has a good understanding of $(X, \overline{X})$-regular singular stratified bundles, one can show that the following general definition of a regular singular stratified bundle is reasonable: A stratified bundle E on X is called regular singular if E is $(X, \overline{X})$-regular singular for all partial compactifications $(X, \overline{X})$ of X. Again we show that the category $\operatorname{Strat}^{\mathrm{rs}}(X)$ of regular singular stratified bundles is a full tannakian subcategory of $\operatorname{Strat}(X)$, and that it is invariant under universal homeomorphisms. We again define exponents as elements of $\mathbb{Z}_p/\mathbb{Z}$, and prove that this definition reduces to the usual one if X admits a good compactification, i.e. a *proper*, smooth k-variety $\overline{X}$, such that $X \subseteq \overline{X}$ is an open subscheme and $\overline{X} \setminus X$ is a strict normal crossings divisor: In this case a stratified bundle E is regular singular if and only if it is $(X, \overline{X})$-regular singular.

In the next section of Chapter 3 we give a discussion of various other notions of regular singularity for stratified bundles. This is inspired by [KS10], but unfortunately the results in our context are far less conclusive. The main result is that if the base field k is uncountable, then a stratified bundle E is regular singular if it is regular singular on all smooth curves in X, and that all notions agree if resolution of singularities is true.

At the end of Chapter 3 we consider the following setup: If E is a stratified bundle on X, ω_0 a k-valued fiber functor for the tannakian subcategory of $\mathrm{Strat}(X)$ spanned by E, and $G := \underline{\mathrm{Aut}}^{\otimes}(\omega_0)$ the monodromy group of E, then tannaka theory canonically associates with E a G-torsor on X, which is smooth by a theorem of J. P. P. dos Santos ([dS07]). This G-torsor is a global version of a Picard-Vessiot extension. We analyze how regular singularity of E is reflected in the properties of the associated torsor.

In Chapter 4 we establish a connection between regular singular stratified bundles with finite monodromy group and tamely ramified coverings. We prove that a stratified bundle on X is regular singular with finite monodromy group if and only if it is trivialized on a finite tamely ramified covering. A main ingredient is the fact that given a good partial compactification $(X, \overline{X})$, a stratified bundle E is $(X, \overline{X})$-regular singular if and only if its exponents are torsion in $\mathbb{Z}_p/\mathbb{Z}$, and regular (i.e. it extends to a stratified bundle on $\overline{X}$) if and only if its exponents are 0 in $\mathbb{Z}_p/\mathbb{Z}$. In particular, there is no analog for a flat connection with nilpotent residues in our setup.

We derive two corollaries: Firstly, if Π_X^{rs} is the pro-algebraic k-group scheme associated with $\mathrm{Strat}^{\mathrm{rs}}(X)$ via tannaka theory, then $\pi_0(\Pi_X^{\mathrm{rs}}) = \pi_1^{\mathrm{tame}}(X)$ considered as a constant k-group scheme. Here we write $\pi_0(G) = G/G^0$ for a k-group scheme G. This is a generalization of a theorem of J. P. P. dos Santos, [dS07]. Secondly, using the results from [KS10], it follows that a stratified bundle with finite monodromy group is regular singular if and only if it is regular singular restricted to all curves on X. The proof of this statement only uses resolution of singularities for surfaces.

In Chapter 5 we give an introduction to Gieseker's conjecture in the non-projective case, and solve it for rank 1 bundles. More precisely, as explained in the introduction, an extension of Gieseker's conjecture to the nonprojective case could be the following: If X is a smooth, separated, finite type k-scheme, then $\pi_1^{\mathrm{tame}}(X) = 0$ if and only if there are no nontrivial regular singular stratified bundles on X. It follows from the results of Chapter 4 that $\pi_1^{\mathrm{tame}}(X) = 0$ if every regular singular stratified bundle is trivial. In the other direction, we show that $\pi_1^{\mathrm{tame}}(X) = 0$ implies that there are no nontrivial stratified line bundles on X. This is a particular case of the above conjecture, since a stratified line bundle is automatically regular singular. The proof does not use resolution of singularities, and its main ingredient is the fact that under these conditions $\mathrm{Pic}\, X$ is a finitely generated abelian group. We prove this using de Jong's theorem on alterations, and simplicial techniques.

In the short final chapter we give an outlook, in which the remaining open questions are collected, and some interesting related problems are posed.

Four appendices containing background information are included for the reader's convenience.

Conventions and notations

(a) The letter k will always denote the base field, which is usually algebraically closed of characteristic $p > 0$.

(b) By Vect_k we denote the category of k-vector spaces, and by Vectf_k the category of finite dimensional vector spaces.

(c) If $\mathcal{C}$ is any category, its Ind-category is denoted by $\mathrm{Ind}(\mathcal{C})$. For example $\mathrm{Ind}(\mathrm{Vectf}_k) = \mathrm{Vect}_k$.

(d) All schemes are assumed to be locally noetherian.

(e) If X is a scheme, then we write $\mathrm{QCoh}(X)$, resp. $\mathrm{Coh}(X)$, resp. $\mathrm{LF}(X)$ for the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules, resp. coherent sheaves of $\mathcal{O}_X$-modules, resp. locally free sheaves of $\mathcal{O}_X$-modules of finite rank.

(f) If X is a k-scheme, then by a *compactification of* X we mean a proper k-scheme Y, such that $X \subseteq Y$ is an open dense subscheme. If X is smooth over k, then we usually write X^N for a normal compactification of X, and we try to reserve $\overline{X}$ for a smooth compactification or a smooth partial compactification, see Appendix A.

(g) If X is a smooth, affine k-scheme, then we say that $x_1, \ldots, x_n \in \Gamma(X, \mathcal{O}_X)$ are *coordinates on* X, if $\Omega^1_{X/k}$ is free on the basis $dx_1, \ldots, dx_n$.

(h) If k is a field of characteristic $p > 0$ and $u : X \to \operatorname{Spec} k$ a k-scheme, then we write F_X for the absolute frobenius of X, and $F^{(n)}_{X/k} : X \to X^{(n)}$ for the n-fold relative frobenius, defined by the diagram

$$\begin{tikzcd}
X \arrow[dr, "F^{(n)}_{X/k}"] \arrow[drr, bend left, "F^n_X"] \arrow[ddr, bend right, "u"'] & & \\
& X^{(n)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "u"] \\
& \operatorname{Spec} k \arrow[r, "F^n_{\operatorname{Spec} k}"] & \operatorname{Spec} k
\end{tikzcd}$$

(i) If X is a connected scheme and $\bar{x} \to X$ a geometric point, we will write $\pi_1(X, \bar{x})$ for the étale fundamental group of X based at $\bar{x}$, and for a profinite group G, we denote by G^{ab}, $G^{(p)}$, $G^{(p')}$ the maximal abelian quotient, resp. the maximal pro-p-quotient, resp. the maximal pro-prime-to-p-quotient.

(j) If G is an affine k-group scheme, then we will denote by $\operatorname{Rep}_k G$ the category of representations of G on k-vector spaces, and by $\operatorname{Repf}_k G$ the category of representations of G on finite dimensional k-vector spaces.

(k) If X is a k-scheme and G an affine group scheme over k, then an X-scheme $h : Y \to X$ is called G-torsor if h is faithfully flat, *affine*, with a G action $Y \times_X G_X \to Y$, such that the induced morphism $Y \times_X G \to Y \times_X Y$ is an isomorphism.

(l) We assume that the reader is familiar with Grothendieck's definition of sheaves of principal parts $\mathcal{P}^n_{X/S}$ ([EGA4, §16]), and of the sheaves of differential operators $\mathscr{D}\!\mathit{iff}^{\,n}_{X/S}(M, N)$ for $\mathcal{O}_X$-modules M, N, for $X \to S$ a morphism of schemes; for an overview in an abstract context we refer to Appendix D. As is customary, we denote by $\mathscr{D}_{X/S}$ the sheaf of differential operators $\mathcal{O}_X \to \mathcal{O}_X$. This is a sheaf of rings, which carries left- and right-$\mathcal{O}_X$-algebra structures.

Chapter 1

Formal local theory of regular singularities

In this chapter we give a detailed overview over the formal local theory of stratified bundles and regular singular stratified bundles. By "formal local" we mean that our base field is a field of Laurent series. In characteristic 0, this situation has been studied in great detail; the relevant terms are "differenial modules" and "differential galois theory". For an overview see e.g. [vdP99]. In positive characteristic, the notion corresponding to a differential module is called "iterative differential module" in [MvdP03].

1.1 Definitions and basic properties

In this section we denote by k a field of positive characteristic p, but we *do not* assume that k is algebraically closed. We write $K := k(\!(x_1, \dots, x_n)\!)$ for the quotient field of $R := k[\![x_1, \dots, x_n]\!]$. All of the results from this section already appear in [Gie75]. Our exposition partly follows [MvdP03, Sec. 1-6], where the case $n = 1$ is treated for arbitrary fields with "iterative derivation".

1.1.1 Definition.

(a) K comes equipped with the sequence of k-linear maps $\partial_{x_i}^{(m)} : K \to K$, for $i = 1, \dots, n$ and $m \geq 0$, such that $\partial_{x_i}^{(0)} = \mathrm{id}_K$ for all i, and for $n > 0$:

$$\partial_{x_i}^{(m)}(x_j^s) = \begin{cases} 0, & i \neq j \\ \binom{s}{m} x_i^{s-m} & \text{else.} \end{cases}$$

(b) The $\partial_{x_i}^{(m)}$ induce k-linear maps $\delta_{x_i}^{(m)} := x_i^m \partial_{x_i}^{(m)}$ on R.

(c) We write $\mathscr{D}_K^{(m)}$ for the K-algebra

$$\mathscr{D}_K^{(m)} := K[\partial_{x_i}^{(r)} | i = 1, \dots, n, r \leq p^m],$$

and $\mathscr{D}_R^{(m)}(\log)$ for the R-algebra

$$\mathscr{D}_R^{(m)}(\log) := R[\delta_{x_i}^{(r)} | i = 1, \dots, n, r \leq p^m].$$

(d) $\mathscr{D}_K := \mathscr{D}_R^{(\infty)} := \varinjlim_m \mathscr{D}_R^{(m)}$.

(e) $\mathscr{D}_R(\log) := \mathscr{D}_R^{(\infty)}(\log) := \varinjlim_m \mathscr{D}_R^{(m)}(\log)$.

(f) We identify $\mathscr{D}_R^{(m)}(\log)$ with a subring of $\mathscr{D}_K^{(m)}$, for $m \in \mathbb{N} \cup \infty$. □

One easily checks the validity of the following Lemma:

1.1.2 Lemma. *For $m \in \mathbb{N} \cup \{\infty\}$ the following relations hold in $\mathscr{D}_K^{(m)}$:*

- *For all $i = 1, \dots, n$ and $f \in K$:*

$$\partial_{x_i}^{(r)} \cdot f = \sum_{\substack{a+b=r \\ a,b \geq 0}} \partial_{x_i}^{(a)}(f) \cdot \partial_{x_i}^{(b)}$$

- *For all $r, s \geq 0$,*

$$\partial_{x_i}^{(r)} \partial_{x_i}^{(s)} = \binom{r+s}{r} \partial_{x_i}^{(r+s)}$$

in $\mathscr{D}_K^{(m)}$, and

$$\delta_{x_i}^{(r)} \delta_{x_i}^{(s)} = \sum_{\substack{a+b=r \\ a,b \geq 0}} \binom{s}{a} \binom{s+b}{s} \delta_{x_i}^{(s+b)}.$$

in $\mathscr{D}_R^{(m)}(\log)$.

- *For all $r, s \geq 0$, and $i, j = 1, \dots n$*

$$\partial_{x_i}^{(r)} \partial_{x_j}^{(s)} = \partial_{x_j}^{(s)} \partial_{x_i}^{(r)},$$

in $\mathscr{D}_K^{(m)}$, and

$$\delta_{x_i}^{(r)} \delta_{x_j}^{(s)} = \delta_{x_j}^{(s)} \delta_{x_i}^{(r)}$$

in $\mathscr{D}_R^{(m)}(\log)$. □

1.1.3 Definition. If E is an $\mathscr{D}_K^{(m)}$-module, and $e \in E$, then e is called *horizontal*, if $\partial_{x_i}^{(r)}(e) = 0$ for $i = 1, \dots, n$ and $r > 0$. The set of horizontal elements is denoted by E^{∇}, and the k-structure of E induces a k-structure on E^{∇}.

An *iterative differential module E of level m*, for $m \in \mathbb{N} \cup \{\infty\}$, on K is a left-$\mathscr{D}_K^{(m)}$-module E which is finite dimensional as a K-vector space. If $m = \infty$, we also say that E is a *stratified bundle* on K.

A *morphism of iterative differential modules* is a morphism of left-$\mathscr{D}_K^{(m)}$-modules. □

1.1.4 Remark. If we consider $K = k(\!(x_1, \dots, x_n)\!)$ as $\mathrm{Frac}(\widehat{\mathcal{O}_{\mathbb{A}_k^n,0}})$, then the situation fits into the general framework of Appendix D, with respect to the formal groupoid $(K, \mathcal{P}_{\mathbb{A}_k^n}^m \otimes K, \dots)$. □

We begin listing the first basic properties of $\mathscr{D}_K^{(m)}$-modules.

1.1.5 Proposition. *Let E, E' be $\mathscr{D}_K^{(m)}$-modules.*

(a) *If E, E' are $\mathscr{D}_K^{(m)}$-modules, then $E \otimes_K E'$ is a $\mathscr{D}_K^{(m)}$-module via*

$$\partial_{x_i}^{(r)}(e \otimes e') := \sum_{\substack{a+b=r\\ a,b\geq 0}} \partial_{x_i}^{(a)}(e) \otimes \partial_{x_i}^{(b)}(e').$$

(b) *If E is an iterative differential module of level ∞, then E^{∇} is a finite dimensional k-vector space and* $\dim_k E^{\nabla} \leq \dim_K E$. □

Proof. Both statements are standard. For the second, note that if e is a horizontal element of E, then eK is a $\mathscr{D}_K^{(m)}$-submodule of E. ■

1.1.6 Proposition. *Let $m < \infty$. The category of iterative differential module of level m is equivalent to the category of finite dimensional $K^{p^{m+1}}$-vector spaces, via the functor:*

$$E \mapsto E_m := \{e \in E | \partial_{x_i}^{(r)}(e) = 0 \textit{ for all } 0 < r \leq p^m\}.$$

More precisely, the canonical morphism $E_m \otimes_{K^{p^{m+1}}} K \to E$ is an isomorphism.

The category of iterative differential module of level ∞ is equivalent to the category of sequences $(E_n, \sigma_n)_{n\geq 0}$, with E_n a finite dimensional K^{p^n}-vector space and σ_n a $K^{p^{n+1}}$-linear map $E_{n+1} \to E_n$, such that

$$K^{p^n} \otimes_{K^{p^{n+1}}} E_{n+1} \xrightarrow{\mathrm{id} \otimes \sigma_n} E_n$$

is an isomorphism. A morphism $\phi : (E_n, \sigma_n) \to (E'_n, \sigma'_n)$ is a sequence of K^{p^n}-linear morphisms $\phi_n : E_n \to E'_n$, such that for every n the squares □

$$\begin{tikzcd}
K^{p^n} \otimes_{K^{p^{n+1}}} E_{n+1} \arrow[r, "\sigma_n"] \arrow[d, "\mathrm{id} \otimes_{K^{p^{n+1}}} \phi_{n+1}"'] & E_n \arrow[d, "\phi_n"] \\
K^{p^n} \otimes_{K^{p^{n+1}}} E'_{n+1} \arrow[r, "\sigma'_n"] & E'_n
\end{tikzcd}$$

Proof. We repeatedly apply Cartier's theorem [Kat70, Thm. 5.1] (although strictly speaking, in [Kat70, Thm. 5.1] it is formulated in a global situation, i.e. for smooth morphisms of schemes, but clearly $k[\![x_1, \ldots, x_n]\!]$ is not smooth.) Since in our setup $(\delta_{x_i}^{(m)})^p = 0$, Cartier's theorem shows that the proposition is true for $m = 1$. If $m > 1$, on E_1 we have the operators $\partial_{x_i}^{(p)}$ acting as order 1 differential operators. So we can apply Cartier's theorem again to see that the theorem is true for $m = 2$. And so forth.

For a global version of this proposition, see [Gie75, Thm. 1.3]. ■

1.1.7 Definition. An iterative differential module M of level ∞ is called *regular singular* if it contains a torsion free R-lattice which is stable under the action of $\mathscr{D}_R^{(\infty)}(\log) \subseteq \mathscr{D}_K^{(\infty)}$. □

1.1.8 Remark. One could define a regular singular iterative differential module of level $m < \infty$ in the same manner, but that would not be an interesting definition: By Proposition 1.1.6, any iterative differential module of level $m < \infty$ admits a basis of horizontal sections, and hence is regular singular.

The notion of a $\mathscr{D}_R^{(m)}(\log)$-module on the other hand *is* interesting (and useful) if $m < \infty$. □

Before we can study regular singular iterative differential modules, we need to know a few facts about congruences for binimial coefficients in characteristic p.

1.1.9 Lemma. *Let p be a prime number.*

(a) Lucas' Theorem: *For $a_0, \ldots, a_n, b_0, \ldots, b_n$ integers in $[0, p-1]$, $a := a_0 + a_1 p + \ldots + a_n p^n$, $b := b_0 + b_1 p + \ldots + b_n p^n$ we have*

$$\binom{a}{b} \equiv \prod_k \binom{a_k}{b_k} \mod p.$$

(b) *If N, n, m, k are integers such that $p^k > n$, then*

$$\binom{N + mp^k}{n} \equiv \binom{N}{n} \mod p.$$

Hence the term $\binom{b}{a} \in \mathbb{Z}/p\mathbb{Z}$ is well-defined for $a, b \in \mathbb{Z}/p^k\mathbb{Z}$, and it can be computed by identifying the set $\mathbb{Z}/p^k\mathbb{Z}$ with the set of integers $\{0, 1, \ldots, p^k - 1\}$. Similarly, $\binom{b}{a}$ is well-defined for $b \in \mathbb{Z}_p$ and $a \in \mathbb{Z}/p^k\mathbb{Z}$.

(c) *If $\alpha \in \mathbb{Z}_p$, then*

$$\alpha = \sum_{n=0}^{\infty} \overline{\binom{\alpha}{p^n}} p^n,$$

where $\overline{a}$ means the unique integer in $[0, \ldots, p-1]$ congruent to a.

(d) *If $\alpha, \beta \in \mathbb{Z}_p$,$d \geq 0$, then*

$$\binom{\alpha\beta}{p^d} \equiv \sum_{\substack{a+b=d \\ a,b \geq 0}} \binom{\alpha}{p^a} \binom{\beta}{p^b} \mod p.$$

Denote by C_d the set of maps of sets $\mathbb{Z}/p^d\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. This is an $\mathbb{F}_p$-algebra via the ring structure on $\mathbb{Z}/p\mathbb{Z}$. For $a \in \mathbb{Z}/p^d\mathbb{Z}$, write $h_a \in C_d$ for the map $b \mapsto \binom{b}{a}$.

(e) *The h_a are a $\mathbb{F}_p$-basis of C_d ([Gie75, Lemma 3.2]).*

(f) *If $p^d < p^m$ then for any $i = 1, \ldots, r$ the assignment (identifying $\mathbb{Z}/p^d\mathbb{Z}$ with $\{0, 1, \ldots, p^d - 1\}$, using (b))*

$$h_a \mapsto \delta_{x_i}^{(a)},$$

gives rise to a well defined morphism $\phi_i : C_d \to \mathscr{D}_R^{(m)}(\log)$ of $\mathbb{F}_p$-algebras, for $m \in \mathbb{N} \cup \{\infty\}$ and $d \leq m$ □

PROOF. (a) This is easily proven by computing the coefficient of x^b of

$$\sum_{k=0}^{a} \binom{a}{k} x^k = (1+x)^a \equiv \prod_{k=0}^{n} (1 + x^{p^k})^{a_k} \mod p.$$

(b) This follows from Lucas' Theorem (a).

(c) First note that for any $k \in \mathbb{Z}_{\geq 0}$ the map $x \mapsto \binom{x}{k}$ is a polynomial map $\mathbb{Z}_p \to \mathbb{Q}_p$ with coefficients in $\mathbb{Q}_p$ and hence continuous. If we write $\alpha = \sum_{n=0}^{\infty} \alpha_n p^n$, then the continuity and (b) imply that

$$\binom{\alpha}{p^n} \equiv \binom{\alpha_{n+1}p^{n+1} + \alpha_n p^n + \ldots + \alpha_0}{p^n} \mod p$$

But by Lucas' Theorem (a) the right hand side is congruent to $\binom{\alpha_n}{1} = \alpha_n$, which proves the claim.

(d) This follows directly from (c).

(e) C_d has p^{d+1} elements, so it suffices to check that the h_a are linearly independent. Assume $\sum_{a \in \mathbb{Z}/p^d\mathbb{Z}} k_a h_a = 0$, with $k_a \in \mathbb{Z}/p\mathbb{Z}$ and some $k_a \neq 0$. Identify the set $\mathbb{Z}/p\mathbb{Z}$ with $\{0, 1, \ldots, p-1\}$. Then $k_0 = 0$, as $\sum_{a \in \mathbb{Z}/p\mathbb{Z}} k_a \binom{0}{a} = k_a$. Assume $k_a = 0$ for every $a \in [0, b]$, $b < p$. Then $\sum_{a \in \mathbb{Z}/p\mathbb{Z}} k_a \binom{b+1}{a} = k_{b+1} = 0$. This proves that the h_a are a basis of C_d.

(f) This follows from the fact that evaluating $\delta_{x_i}^{(a)}$ at x_i^b precisely gives $\binom{b}{a} x_i^b$. ∎

From this we can infer:

1.1.10 Proposition. *For the ring $\mathscr{D}_R^{(m)}(\log) \subseteq \mathscr{D}_K^{(m)}$, $m \in \mathbb{N} \cup \{\infty\}$, the following statements are true:*

(a) *The $\delta_{x_i}^{(r)}$ commute.*

(b) $(\delta_{x_i}^{(r)})^p = \delta_{x_i}^{(r)}$.

(c) ([Gie75, Lemma 3.12]) *Every iterative differential module of level ∞, which is 1-dimensional as a K-vector space, is regular singular.* □

PROOF. (a) and (b) follow from the composition rules in Lemma 1.1.2. By the definition of $\mathscr{D}_K^{(\infty)}$, a left-$\mathscr{D}_K^{(\infty)}$-action on K is determined by giving $\partial_{x_i}^{(r)}(1)$, for $r \geq 1$ and $i = 1, \ldots, n$. Over K, this is equivalent to giving $\delta_{x_i}^{(p^r)}(1)$ for $r \geq 1$ and $i = 1, \ldots, n$. By Proposition 1.1.6, for every $m \geq 0$, there exists a nonzero $f \in K$, such that $\delta_{x_i}^{(p^r)}(f \cdot 1) = 0$ for all $i = 1, \ldots, n$ and $r \leq m$. But then $\delta_{x_i}^{(p^r)}(1) = \delta_{x_i}^{(p^r)}(f^{-1}) f \cdot 1$. But $\delta_{x_i}^{(p^r)}(f^{-1}) f \in R$, so the iterative connection is regular singular. ∎

1.1.11 Definition. Given $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}/p^{m+1}\mathbb{Z}$ (resp. $\in \mathbb{Z}_p$) write

$$\mathcal{O}_R(\alpha_1, \ldots, \alpha_n)$$

for the $\mathscr{D}_R^{(m)}(\log)$-module (resp. $\mathscr{D}_R^{(\infty)}$-module) of rank 1 over R given by

$$\delta_{x_i}^{(r)}(1) = \binom{\alpha_i}{r}.$$

Similarly, we write

$$\mathcal{O}_K(\alpha_1, \ldots, \alpha_n)$$

for the $\mathscr{D}_K^{(m)}$-left-module

$$\mathcal{O}_R(\alpha_1, \ldots, \alpha_n) \otimes_R K.$$ □

1.1.12 Proposition ([Gie75, Thm. 3.3]). *Let E be a left-$\mathscr{D}_R^{(m)}(\log)$-module, and assume that E is free of finite rank as an R-module. Then there is an isomorphism of $\mathscr{D}_R^{(m)}(\log)$-modules*

$$E \cong \bigoplus_{j=1}^{\operatorname{rank} E} \mathcal{O}_R(\alpha_{1,j}, \dots \alpha_{n,j}) \tag{1.1}$$

for some $\alpha_{ij} \in \mathbb{Z}/p^{m+1}\mathbb{Z}$ if $m < \infty$ or $\alpha_{ij} \in \mathbb{Z}_p$ if $m = \infty$. □

PROOF. First fix some $m < \infty$. For notational convenience, we write $E_s := E/(x_i^s)$. Let C_{m+1} and $h_b \in C_{m+1}$ be defined as in Lemma 1.1.9. For every s and for every i, we can define a map $\phi_{s,i}^{(m)} : C_{m+1} \to \mathscr{D}_R^{(m)}(\log) \to \operatorname{End}_k(E_s)$, by sending h_b to $\nabla(\delta_{x_i}^{(b)}) \otimes R/(x_i^s)$, since the operators $\delta_{x_i}^{(a)}$ act on E_s. By Lemma 1.1.9 the map $\phi_{s,i}^{(m)}$ is in fact a morphism of $\mathbb{F}_p$-algebras. For $\alpha \in \mathbb{Z}/p^{m+1}\mathbb{Z}$, write $\chi_\alpha \in C_{m+1}$ for the characteristic function of α defined by

$$\chi_\alpha(b) = \begin{cases} 0 & b \neq \alpha \\ 1 & \text{else.} \end{cases}$$

The set $\{\chi_\alpha | \alpha \in \mathbb{Z}/p^{m+1}\mathbb{Z}\}$ is a set of orthognal idempotents in C_{m+1}, and the set of their images $\phi_{s,i}^{(m+1)}(\chi_\alpha) \in \operatorname{End}_k(E_s)$ is also a set of orthognal idempotents. Thus we obtain a decomposition

$$E_s = \bigoplus_{\alpha \in \mathbb{Z}/p^{m+1}\mathbb{Z}} V_{i,s,\alpha}, \tag{1.2}$$

such that each χ_α acts as the identity on $V_{i,s,\alpha}$ and trivially on $V_{i,s,\beta}$ if $\beta \neq \alpha$. Since for $b \in \mathbb{Z}/p^{m+1}\mathbb{Z}$ we can write $h_b = \sum_{\alpha \in \mathbb{Z}/p^{m+1}\mathbb{Z}} \binom{\alpha}{b} \chi_\alpha$, the operator $\phi_{s,i}^{(m+1)}(h_b) = \delta_{x_i}^{(b)}$ acts on $V_{i,s,\alpha}$ by multiplication with the scalar $\binom{\alpha}{b}$.

Since $E_{s+1} = E_s \oplus x_i E_s$ as k-vector space, we get a decomposition

$$\begin{aligned} E_{s+1} &= \left(\bigoplus_{\alpha \in \mathbb{Z}/p^{m+1}} V_{i,s,\alpha} \right) \bigoplus \left(\bigoplus_{\alpha \in \mathbb{Z}/p^{m+1}} x_i V_{i,s,\alpha} \right) \\ &= \bigoplus_{\alpha \in \mathbb{Z}/p^{m+1}} V_{i,s+1,\alpha} \end{aligned}$$

It is easy to see that $\delta_{x_i}^{(b)}$ acts on $x_i V_{i,s,\alpha}$ via $\binom{\alpha+1}{b}$, so $V_{i,s+1,\alpha} = V_{i,s,\alpha} \oplus x_i V_{i,s,\alpha-1}$, and the reduction map $E_{s+1} \twoheadrightarrow E_s$ maps $V_{i,s+1,\alpha}$ surjectively onto $V_{i,s,\alpha}$. Thus, passing to the limit over s we obtain a decomposition

$$E = \bigoplus_{\alpha \in \mathbb{Z}/p^{m+1}} V_{i,\alpha}$$

such that h_b acts on each $V_{i,\alpha}$ as multiplication by $\binom{\alpha}{b}$.

Finally, we can do this construction for $i = 1, \dots, r$, and since the operators $\delta_{x_i}^{(a)}$ and $\delta_{x_j}^{(b)}$ commute for all i, j, a, b, we can find a basis $f_1, \dots, f_{\operatorname{rank} E}$ of $E \otimes R/(x_1, \dots, x_n)$, such that each $\delta_{x_i}^{(a)}$, $i = 1, \dots, r$, $a \in \{0, 1, \dots, p^{m+1} - 1\}$

acts on f_j by a scalar multiplication with $\binom{\alpha_{i,j}}{a}$, for $\alpha_{i,1}, \ldots, \alpha_{i,r} \in \mathbb{Z}/p^{m+1}$. Since E was assumed to be free, we can lift this basis to a basis of the free R-module E, and finally get the decomposition

$$E = \bigoplus_{j=1}^{\operatorname{rank} E} \mathcal{O}(\alpha_{1,j}, \alpha_{2,j}, \ldots, \alpha_{r,j}),$$

so the proof is complete in the case that $m < \infty$.

Now we have to let m vary. We also have a map $\phi_{i,s}^{(m)} : C_m \to \operatorname{End}_k(E_s)$, and we get a commutative triangle

$$\begin{tikzcd} C_m \arrow[dd, "\rho"'] \arrow[dr, "\phi_{i,s}^{(m)}"] & \\ & \mathscr{E}nd_k(E_s) \\ C_{m+1} \arrow[ur, "\phi_{i,s}^{(m+1)}"'] & \end{tikzcd}$$

where $\rho : C_m \to C_{m+1}$ is simply the composition with reduction modulo p^m. Then for $\alpha_m \in \mathbb{Z}/p^m\mathbb{Z}$, we have

$$\rho(\chi_{\alpha_m}) = \sum_{\substack{\beta \in \mathbb{Z}/p^{m+1} \\ \beta \equiv \alpha_m \mod p^m}} \chi_\beta$$

It follows that the decomposition

$$E_s = \bigoplus_{\alpha_{m+1} \in Z/p^{m+1}} V_{i,s,\alpha_{m+1}}$$

refines the decomposition (1.2). Since E is of finite rank, this process stabilizes for $m \to \infty$, and we obtain a decomposition

$$E_s = \bigoplus_{\alpha \in \mathbb{Z}_p} V_{i,s,\alpha} \tag{1.3}$$

such that $\delta_{x_j}^{(a)}$ acts via multiplication by $\binom{\alpha}{a}$ on $V_{i,s,\alpha}$. The same arguments as before allow us to first pass to the limit over s, then to do the process for $i = 1, \ldots, r$, and then to find a basis of E giving the common decomposition

$$E \cong \bigoplus_{j=1}^{\operatorname{rank} E} \mathcal{O}_R(\alpha_{1,j}, \ldots, \alpha_{n,j})$$

with $\alpha_{ij} \in \mathbb{Z}_p$ ■

1.1.13 Corollary. *An iterative differential module of level ∞ is regular singular if and only if it is a direct sum of iterative differential modules of rank 1, and any iterative differential module of rank 1 is isomorphic to $\mathcal{O}_K(\alpha)$ for some $\alpha \in \mathbb{Z}_p$.* □

PROOF. By Proposition 1.1.10, an iterative differential module of level ∞ and of rank 1 is regular singular. Clearly, a direct sum of regular singular iterative differential modules of level ∞ is regular singular, so the corollary follows directly from Proposition 1.1.12. ■

1.1.14 Corollary ([MvdP03, Prop. 6.3]). *If E is an iterative differential module of level $m \in \mathbb{N} \cup \{\infty\}$ and rank > 1, then E contains a nontrivial iterative differential submodule of level m.* □

PROOF. The proof is the same as in [MvdP03, Prop. 6.3] although there it is only stated for $n = 1$; we include it for completeness.

For $m < \infty$, there exists a horizontal basis of E by Proposition 1.1.6, so we are done.

For $m = \infty$, the goal is to produce a nontrivial regular singular submodule of E and then apply Proposition 1.1.12.

Let E_0 be any R-sublattice of E, and define $E_0^{(m)}$ to be the subset

$$E_0^{(m)} := \{e \in E_0 | \delta_{x_i}^{(r)}(e) \in E_0, i = 1, \dots, n, 0 < r \leq p^m\}$$

Then $E_0^{(\infty)} := \bigcap_{m \geq 0} E_0^{(m)}$ is a $\mathscr{D}_R^{(\infty)}(\log)$-stable sublattice of E. We have to show that $E_0^{(\infty)} \neq 0$.

Let $\mathfrak{m}$ be the maximal ideal of R, and r_m the minimal integer such that $\mathfrak{m}^{r_m} E_0^{(m)} \subseteq E_0$. Then $\mathfrak{m}^{r_m} E_0^{(m)} \not\subseteq \mathfrak{m} E_0$, and for $e \in \mathfrak{m}^{r_m} E_0^{(m)} \setminus \mathfrak{m} E_0$, we have $\delta_{x_i}^{(a)}(e) \in \mathfrak{m}^{r_m} E_0^{(m)} \subseteq E_0$ for $a \leq p^m$, so $e \in E_0^{(m)}$, and hence $E_0^{(m)} \not\subseteq \mathfrak{m} E_0$.

Since R is complete in the $\mathfrak{m}$-adic topology, it follows that $E_0^{(\infty)} \neq 0$. ■

1.1.15 Definition. If E is a left-$\mathscr{D}_R^{(m)}$-module which is free of finite rank as an R-module, then the numbers α_{ij} from (1.1) are called *exponents of E along x_i*. If E is torsion free (but perhaps not locally free), then the exponents of E are defined to be the *exponents of E along x_i* are the exponents of

$$E \otimes k(\!(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)\!)[\![x_i]\!]$$

along x_i. □

1.1.16 Remark. Definition 1.1.15 is the reason why we had to drop the assumption that k is algebraically closed (or even perfect) in this section. Otherwise we could not appeal to a decomposition over

$$k(\!(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)\!)[\![x_i]\!].$$

This is not explicitly mentioned in [Gie75]. □

Gieseker also proves the following important result:

1.1.17 Proposition ([Gie75, Thm. 3.5]). *Let E be a $\mathscr{D}_R^{(\infty)}(\log)$-module and assume that E is a torsion free R-module of finite rank satisfying Serre's condition (S_2), i.e. for every point $P \in \operatorname{Spec} R$ of codimension ≥ 2, we have $\operatorname{depth}_P E \geq 2$. Then E is free, if the set of exponents injects into $\mathbb{Z}_p/\mathbb{Z}$.* □

1.2 Fiber functors and Picard-Vessiot theory

We continue denote by k a field of characteristic $p > 0$, $R = k[\![x_1, \dots, x_n]\!]$, $K = k(\!(x_1, \dots, x_n)\!)$. *From now on we assume that k is algebraically closed.*

1.2.1 Definition. We write $\mathrm{Strat}(K)$ for the category of iterative differential modules of level ∞ (which we also called stratified bundles) on K, and $\mathrm{Strat}^{\mathrm{rs}}(K)$ for the full subcategory of regular singular iterative differential modules of level ∞. ◻

For the notions from the theory of tensor categories and tannakian categories we refer to Appendix C.

1.2.2 Proposition. *The category $\mathrm{Strat}(K)$ is a tensor category over k (Definition C.1.1, (f)), and $\mathrm{Strat}^{\mathrm{rs}}(K)$ is a tensor subcategory (Definition C.1.5) over k, with identity object the 1-dimensional K vector space $\mathbb{I}_K$ with the canonical action of $\mathscr{D}_K^{(\infty)}$.*

If $E \in \mathrm{Strat}(K)$, then the full tensor subcategory $\langle E \rangle_{\otimes} \subseteq \mathrm{Strat}(K)$ generated by E is a neutral tannakian category. If $\iota : \mathrm{Strat}^{\mathrm{rs}}(K) \hookrightarrow \mathrm{Strat}(X)$ is the inclusion functor, then for $E \in \mathrm{Strat}^{\mathrm{rs}}(K)$, ι restricts to an equivalence $\langle E \rangle_{\otimes} \xrightarrow{\sim} \langle \iota(E) \rangle_{\otimes}$. ◻

PROOF. The first part is standard: If E, E', F, F' are $\mathscr{D}_K^{(\infty)}$-modules, one defines $\mathscr{D}_K^{(\infty)}$-structures on $E \otimes_K E'$ and $\mathrm{Hom}_{\mathscr{D}_K^{(\infty)}}(E, E')$, such that $E \otimes_K \mathbb{I} = E$, $\mathrm{End}(\mathbb{I}_K) = k$, $(E^{\vee})^{\vee} \cong E$ and

$$\mathrm{Hom}_{\mathscr{D}_K^{(\infty)}}(E, E') \otimes \mathrm{Hom}_{\mathscr{D}_K^{(\infty)}}(F, F') \cong \mathrm{Hom}_{\mathscr{D}_K^{(\infty)}}(E \otimes F, E \otimes F'),$$

i.e. $\mathrm{Strat}(K)$ is a rigid k-linear symmetric monoidal category. It is also abelian: If $\phi : E \to E'$ is a $\mathscr{D}_K^{(\infty)}$-linear map, then the kernel and cokernel of ϕ in Vect_K can canonically be equipped with a $\mathscr{D}_K^{(\infty)}$-structure. It is also clear that $\mathrm{End}(\mathbb{I}_K) = k$.

Moreover, if E, E' are regular singular, then so are the stratified bundles $E \otimes_K E'$ and $\mathrm{Hom}_{\mathscr{D}_K^{(\infty)}}(E, E')$: One just has to take the tensor product, resp. the Hom of two $\mathscr{D}_R^{(\infty)}(\log)$-stable lattices.

If E is regular singular and $E' \subseteq E$ an iterative differential submodule of level ∞, then E' is also regular singular: If E_0 is a $\mathscr{D}_R^{(\infty)}(\log)$-stable R-submodule of E, then $E_0 \cap E'$ is a $\mathscr{D}_R^{(\infty)}(\log)$-stable R-submodule of E'. Finally, If $E' \subseteq E$ are two regular singular stratified bundles, then their quotient is also regular singular: A $\mathscr{D}_R^{(\infty)}(\log)$-stable lattice for E/E' is $E_0/(E_0 \cap E')$. This proves that $\langle E \rangle_{\otimes}$ is equivalent to $\langle \iota E \rangle_{\otimes}$, and thus that $\mathrm{Strat}^{\mathrm{rs}}(K) \subseteq \mathrm{Strat}(K)$ is a $\otimes$-subcategory stable under Hom, $\otimes$, $\oplus$ and subquotients.

Finally, since k is algebraically closed, $\langle E \rangle_{\otimes}$ admits a k-valued fiber functor by Deligne's theorem Theorem C.1.7. ■

In the rest of this section we recall and generalize [Del90, Sec. 9]:

1.2.3 Definition. First note that K together with its canonical $\mathscr{D}_K^{(\infty)}$-structure is a ring in the category of $\mathscr{D}_K^{(\infty)}$-modules. A K-algebra A with $\mathscr{D}_K^{(\infty)}$-structure is called *differential K-algebra* if it is an algebra over the $\mathscr{D}_K^{(\infty)}$-ring K in the category of $\mathscr{D}_K^{(\infty)}$-modules, i.e. its $\mathscr{D}_K^{(\infty)}$-structure extends the canonical structure of K, and if multiplication $A \otimes A \to A$ is a morphism of $\mathscr{D}_K^{(\infty)}$-modules. In other words: For $x, y \in A$,

$$\partial_{x_i}^{(m)}(xy) = \sum_{\substack{a+b=m \\ a,b \geq 0}} \partial_{x_i}^{(a)}(x) \partial_{x_i}^{(b)}(y)$$

For a more general context, see also Definition C.3.4. □

1.2.4 Definition. Let $E \in \mathrm{Strat}(K)$. A K-algebra A is called *Picard-Vessiot ring for E*, if

(a) A carries the structure of a differential K-algebra.

(b) A contains no nontrivial ideals stable under the $\mathscr{D}_K^{(\infty)}$-action.

(c) The $\mathscr{D}_K^{(\infty)}$-module $E \otimes_K A$ is generated by horizontal elements.

(d) If $e_1, \dots, e_d$ is a basis of of E, and $f_1, \dots, f_d$ a horizontal basis of $E \otimes_K A$, then A is generated by the coefficients of the associated change of basis matrix F and $1/\det F$.

A field L with $K \subseteq L$ and a $\mathscr{D}_K^{(\infty)}$-action is called *Picard-Vessiot extension for E*, if $L = \mathrm{Frac}\, A$ for a Picard-Vessiot ring A, such that the $\mathscr{D}_K^{(\infty)}$-action of L extends the one of A. □

1.2.5 Example. Let E be a stratified bundle on K. We explicitly construct a Picard-Vessiot ring for E. If $K = k(\!(t)\!)$, then this is [MvdP03, Lemma 3.4]. Let $e_1, \dots, e_d$ a basis for E over K. First lets try to rewrite the $\mathscr{D}_K^{(\infty)}$-action as a set of differential equations for elements of K:

We can write $\partial_{x_i}^{(m)}(e_j) = \sum_{k=1}^d a_{i,k,j}^{(m)} e_k$. Write $A_i^{(m)} = (a_{i,k,j}^{(m)})_{k,j}$. Now if $y = \sum_{i=1}^d y_i e_i \in E$, then

$$\partial_{x_i}^{(m)}(y) = (\partial_{x_i}^{(m)}(y_j))_j + A_i^{(1)}(\partial_{x_i}^{(m-1)}(y_j))_j + A_i^{(2)}(\partial_{x_i}^{(m-2)}(y_j))_j + \dots + A_i^{(m)} y, \tag{1.4}$$

where $(\partial_{x_i}^{(m)}(y_j))_j$ is considered as a column vector.

We want conditions for $y \in E$ to be horizontal, i.e. for $\partial_{x_i}^{(m)}(y) = 0$ for all i, m. Define $B_i^{(1)} := -A_i^{(1)}$. Then $\partial_{x_i}^{(1)}(y) = 0$ if and only if we have an equality of vectors $(\partial_{x_i}^{(1)}(y_j))_j = B_i^{(1)} y$. Assume we have found matrices $B_i^{(r)}$, $r = 1, \dots, m-1$, such that $\partial_{x_i}^{(r)}(y) = 0$ for $r = 1, \dots, m-1$ if and only if $(\partial_{x_i}^{(r)}(y_j))_j = B_i^{(r)} y$ for all $r = 1, \dots, m-1$. Then we see from (1.4) that $\partial_{x_i}^{(r)}(y) = 0$ for $r = 1, \dots, m$ if and only if

$$(\partial_{x_i}^{(m)}(y_j))_j = -\left(A_i^{(1)} B_i^{(m-1)} + A_i^{(2)} B_i^{(m-2)} + \dots + A_i^{(m)}\right) y, \tag{1.5}$$

so we can define $B_i^{(m)}$ via the right hand side of (1.5).

To summarize, we have found matrices $B_i^{(m)}$, $i = 1, \dots, n$, $m \geq 0$, such that $y = y_1 e_1 + \dots + y_d e_d \in E^\nabla$ if and only if $(\partial_{x_i}^{(m)}(y_j))_j = B_i^{(m)} y$ for all m, i.

Assume that there is a matrix $F = (f_{rs}) \in \mathrm{GL}_d(K)$, such that there is an equality of matrices

$$(\partial_{x_i}^{(m)}(f_{rs}))_{r,s} = B_i^{(m)} F \text{ for all } i = 1, \dots, n \text{ and } m \geq 0. \tag{1.6}$$

Then, by construction, $Fe_1, \dots, Fe_d$ is a basis of horizontal sections.

Finally, to construct a Picard-Vessiot ring for E, we "formally add" the coefficients of such a matrix F to K.

More precisely: Let $R_0 := K[X_{ij}, (\det(X_{ij}))^{-1} | 1 \leq i, j \leq d]$, and define $\partial_{x_i}^{(m)}(X_{rs})$ via the matrix equation

$$\left(\partial_i^{(m)}(X_{rs})\right)_{0 \leq r,s \leq d} := B_i^{(m)} \cdot (X_{rs})_{0 \leq r,s \leq d} \,.$$

Since the $B_i^{(m)}$ come from a $\mathscr{D}_K^{(\infty)}$-action, this defines a $\mathscr{D}_K^{(\infty)}$-action on R_0, extending the action on K, and such that R_0 is a differential K-algebra. Clearly, $E \otimes_K R_0$ admits a basis of horizontal sections. Now let I be an ideal of R_0, maximal among the ideals stable under the $\mathscr{D}_K^{(\infty)}$-action, then it follows immediately that R/I is a Picard-Vessiot ring for E. □

We now connect the notions of Picard-Vessiot rings and fields with the tannakian theory of the category $\langle E \rangle_\otimes$.

Recall that for a $\mathscr{D}_K^{(\infty)}$-module M, we write M^∇ for the k-vector space of horizontal elements.

1.2.6 Proposition (Compare to [Del90, §.9]). *Let E be a stratified bundle on K and $\rho_{\mathrm{forget}} : \langle E \rangle_\otimes \to \mathrm{Vectf}_K$ the forgetful functor.*

(a) *If A is a Picard-Vessiot ring for E, then the functor*

$$E' \mapsto (E' \otimes_K A)^\nabla$$

is a fiber functor $\omega_A : \langle E \rangle_\otimes \to \mathrm{Vectf}_k$.

(b) *If $\omega_0 : \langle E \rangle_\otimes \to \mathrm{Vectf}_k$ is a fiber functor, and $G := \underline{\mathrm{Aut}}^\otimes(\omega_0)$, then the G_K-torsor $\underline{\mathrm{Isom}}_K^\otimes(\omega_0 \otimes_k K, \rho_{\mathrm{forget}})$ is representable by a K-algebra A_{ω_0} which is a Picard-Vessiot ring for E. There is a canonical isomorphism of functors $\omega_0 \cong \omega_{A_{\omega_0}}$.*

(c) *These two constructions set up a correspondence*

$$\left\{\textit{fiber functors } \langle E \rangle_\otimes \to \mathrm{Vectf}_k\right\} \leftrightarrow \{\textit{Picard-Vessiot rings for } E\} \,. \qquad \square$$

Proof. This is an instance of general statement about tannakian categories; see the discussion after Definition C.3.4.

(a) Since $E \otimes_K A$ admits a basis of horizontal elements, so does $E' \otimes_K A$ for every $E' \in \langle E \rangle_\otimes$. Hence for $E' \in \langle E' \rangle_\otimes$, $(E' \otimes_K A)^\nabla \otimes_k A = E' \otimes_K A$, which shows that the functor $E' \mapsto E' \otimes_K A \mapsto (E' \otimes_K A)^\nabla$ is a faithful, exact, k-linear $\otimes$-functor and hence a fiber functor.

(b) This is essentially [Del90, Prop. 9.3], although there it is only written for $\mathrm{char}(k) = 0$ and $n = 1$. We check that the argument works for arbitrary n and $\mathrm{char}(k) = p > 0$.

The G_K-torsor $P_{\omega_0} := \underline{\mathrm{Isom}}_K^{\otimes}(\omega_0 \otimes_k K, \rho_{\mathrm{forget}})$ is representable by a K-algebra A_{ω_0} according to Theorem C.3.2. By the main theorem of the tannaka formalism, the functor functor ω_0 induces an equivalence

$$\omega_0' : \mathrm{Ind}(\langle E \rangle_{\otimes}) \xrightarrow{\sim} \mathrm{Ind}(\mathrm{Repf}_k\, G),$$

where $\mathrm{Ind}(-)$ denotes the category of inductive systems. Note that by Proposition C.2.2, $\mathrm{Ind}(\mathrm{Repf}_k\, G) \cong \mathrm{Rep}_k\, G$.

The finite type k-group scheme G is affine. If we write $G = \mathrm{Spec}\, \mathcal{O}_G$, then $\mathcal{O}_G$, together with the comultiplication $\Delta : \mathcal{O}_G \to \mathcal{O}_G \otimes_k \mathcal{O}_G$ is the right regular representation, i.e. it corresponds to G acting on itself from the right. Note that $(\mathcal{O}_G, \Delta)$ is a commutative algebra over the unit object in the category $\mathrm{Rep}_k\, G$.

The forgetful functor ρ_{forget} also extends to Ind-categories, and it follows from Theorem C.3.2 that there is a unique (up to unique isomorphism) algebra A_{ω_0} over the trivial stratified bundle K in $\mathrm{Ind}(\langle E \rangle_{\otimes})$, such that $P_{\omega_0} = \underline{\mathrm{Isom}}_K^{\otimes}(\omega_0 \otimes_k K, \rho_{\mathrm{forget}})$ is represented by $\rho_{\mathrm{forget}}(A_{\omega_0})$, and such that $\omega_0'(A_{\omega_0}) = (\mathcal{O}_G, \Delta)$. In particular, A_{ω_0} is a differential K-algebra (Definition 1.2.3) and we show that A_{ω_0} is a Picard-Vessiot ring for E.

Note that under the functor ω_0', the sub-$\mathscr{D}_K^{(\infty)}$-modules of A_{ω_0} correspond to the subrepresentations of $(\mathcal{O}_G, \Delta)$. In particular, $A_{\omega_0}^{\nabla} = (\mathcal{O}_G, \Delta)^G = k$. Similarly, the $\mathscr{D}_K^{(\infty)}$-invariant ideals in A_{ω_0} correspond to closed subschemes of G, which are invariant under translation. But there are no nontrivial such subschemes, so A_{ω_0} has no nontrivial ideals stable under the $\mathscr{D}_K^{(\infty)}$-action. This shows that Definition 1.2.4, (b) holds.

For Definition 1.2.4, (c), denote by

$$\nu_1, \nu_2 : \langle E \rangle_{\otimes} \to \mathrm{Ind}(\langle E \rangle_{\otimes}),$$

the functors defined by

$$\nu_1(E') := \omega_0(E') \otimes_k A_{\omega_0},\ \nu_2(E') := E' \otimes_k A_{\omega_0}.$$

Then by applying ω_0' to Proposition C.2.6, we see that there is a natural isomorphism of functors $\nu_1 \cong \nu_2$. Composing with $(-)^{\nabla}$ gives an isomorphism of functors $\omega_0 \cong \omega_{A_{\omega_0}}$, which means that A_{ω_0} satisfies Definition 1.2.4, (c).

Finally, for Definition 1.2.4, (d), let $e_1, \ldots, e_d$ be a basis for E. A trivialization of $E \otimes_K A_{\omega_0}$ over A_{ω_0} gives an element

$$P_{\omega_0}(P_{\omega_0}) = \underline{\mathrm{Isom}}_K^{\otimes}(\omega_0 \otimes_k K, \rho_{\mathrm{forget}})(A_{\omega_0}) = \mathrm{Hom}(A_{\omega_0}, A_{\omega_0}).$$

Since $\underline{\mathrm{Isom}}_K^{\otimes}(\omega_0 \otimes_k K, \rho_{\mathrm{forget}}) \otimes_K A_{\omega_0}$ is the trivial $G_{A_{\omega_0}}$-torsor, we may assume (by translating if necessary) that $\mathrm{id} \in P_{\omega_0}(P_{\omega_0})$ corresponds to a trivialization of $E \otimes_K A_{\omega_0}$, i.e. to an isomorphism of $\mathscr{D}_K^{(\infty)}$-modules $\Phi : \omega_0(E) \otimes_k A_{\omega_0} \to E \otimes_K A_{\omega_0}$. Define $f_i := \Phi(e_i)$. Then $f_1, \ldots, f_d$ is a horizontal basis for $E \otimes_K A_{\omega_0}$ and we can consider Φ as the matrix changing the basis e_i to the basis f_i. Define A' to be the subalgebra of A_{ω_0} generated by the coefficients of the change of base matrix $\Phi := (\Phi_{ij})_{i,j}$ and $1/\det\Phi$. Note that A' is stable under the $\mathscr{D}_K^{(\infty)}$-action of A_{ω_0}: There are matrices $B_{n,i}$ with entries in K, such that $(\partial_{x_i}^{(n)}\Phi_{ij})_{i,j} = B_{n,i}\Phi$, see Example 1.2.5. Then Φ descends to an isomorphism of $\mathscr{D}_K^{(\infty)}$-modules $\Phi' : \omega_0(E) \otimes_k A' \xrightarrow{\sim} E \otimes_k A'$, and thus to a morphism $\alpha_{\Phi'} : A_{\omega_0} \to A'$, arising from

$$\operatorname{Spec} A' \to \underline{\mathrm{Isom}}_K^{\otimes}(\omega_0 \otimes_k K, \rho_{\mathrm{forget}}). \qquad \blacksquare$$

But by construction, the composition $A_{\omega_0} \xrightarrow{\alpha_{\Phi'}} A' \hookrightarrow A_{\omega_0}$ is the identity, so $A_{\omega_0} = A'$, and we are done.

(c) Let $\omega_0 : \langle E\rangle_{\otimes} \to \mathrm{Vectf}_k$ be a fiber functor. We just constructed a Picard-Vessiot ring A_{ω_0} for E, such that ω_0 is canonically isomorphic to the functor $\omega_{A_{\omega_0}}(-) := (- \otimes_K A_{\omega_0})^{\nabla}$.

Conversely, if A is a Picard-Vessiot ring for E, then we saw that ω_A is a fiber functor, so it remains to see that $P_A := \underline{\mathrm{Isom}}_K^{\otimes}(\omega_A \otimes_k K, \rho_{\mathrm{forget}}) = \operatorname{Spec} A$. But this is easy: Let $P_A = \operatorname{Spec} B$. Then B is a Picard-Vessiot ring, and since $E \otimes_K A$ is trivializable, we get a morphism $B \to A$. This morphism is injective, since B does not have any $\mathscr{D}_K^{(\infty)}$ stable ideals, and it is surjective, since B and A are generated over K by the same elements.

1.2.7 Corollary. *If E is a stratified bundle on K, then all Picard-Vessiot rings for E are (non-canonically) isomorphic.* $\square$

Proof. By Proposition 1.2.6, we have seen that the Picard-Vessiot rings give rise to, and are determined by, fiber functors $\langle E\rangle_{\otimes} \to \mathrm{Vectf}_k$. If ω_1, ω_2 are two such fiber functors, then $\underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_2)$ is a $\underline{\mathrm{Aut}}^{\otimes}(\omega_1)$-torsor on k. But k is algebraically closed by assumption, so every torsor over k is trivial, and thus $\omega_1 \cong \omega_2$. $\blacksquare$

1.2.8 Corollary. *If $E \in \mathrm{Strat}(K)$, then E is regular singular, if and only if a Picard-Vessiot ring A for E is an increasing union $\bigcup_i E_i$ with $E_i \in \mathrm{Strat}^{\mathrm{rs}}(K)$.* $\square$

Proof. A Picard-Vessiot ring A gives a fiber functor $\omega_A : \langle E\rangle_{\otimes} \to \mathrm{Vectf}_k$. Let $G := \underline{\mathrm{Aut}}^{\otimes}(\omega_A)$. The functor ω_A gives an equivalence $\omega'_A : \mathrm{Ind}(\langle E\rangle_{\otimes}) \to \mathrm{Ind}(\mathrm{Repf}_k G)$, where $\mathrm{Ind}(-)$ denotes the category of inductive systems. Note that $\mathrm{Ind}(\mathrm{Repf}_k G) = \mathrm{Rep}_k G$ by Proposition C.2.2. Moreover, by construction $\omega'_A(A) = \mathcal{O}_G$, the regular representation of G, i.e. $G = \operatorname{Spec}\mathcal{O}_G$. This shows that $\mathrm{Ind}(\langle E\rangle_{\otimes}) \cong \langle A\rangle_{\otimes}$, since any representation of G is a subrepresentation of $\mathcal{O}_G^n$ for some n.

If E is regular singular, then every object of $\langle E\rangle_{\otimes}$ is regular singular, so A is a increasing union of regular singular objects.

Conversely, since $E \subseteq A^n$ for some n, E is regular singular if A is an increasing union of regular singular objects. $\blacksquare$

1.2.9 Remark. For a similar statement as Corollary 1.2.8 in a global context, see Section 3.5. $\square$

1.3 Differential Galois groups

Classically, the differential galois group associated with a differential module E is the the group of differential autmorphisms of its Picard-Vessiot extension. By Proposition 1.2.6, this notion is intimately related to tannaka theory. In this section, we develop this relation, and discuss some properties of differential galois groups.

We continue to denote by k an algebraically closed field of characteristic $p > 0$.

1.3.1 Definition. Let E be a stratified bundle over K, and A a Picard-Vessiot ring for E. The *differential galois group G_A of E with respect to A* is the group of K-algebra automorphisms of A which are morphisms of $\mathscr{D}_K^{(\infty)}$-modules.

Since $E \otimes_K A$ admits a basis of horizontal sections, G_A acts faithfully on $(E \otimes_K A_E)^\nabla$, so G_A can be identified with a reduced algebraic subgroup of $\mathrm{GL}_k((E \otimes_K A_E)^\nabla)$. □

Since the construction of Picard-Vessiot rings in Example 1.2.5 is fairly explicit, the differential galois group can be computed in many cases. Before doing this, we want to connect the differential galois group with the tannakian fundamental group of a stratified bundle E. First a lemma:

1.3.2 Lemma. *Let E be a stratified bundle on K. A Picard-Vessiot ring for E is an integral domain.* □

PROOF. For $n = 1$, this is [MvdP03, Lemma 3.2]. The proof works almost without change: Let A be a Picard Vessiot ring, and define $\phi : A \to A[\![T_1, \dots, T_n]\!]$ by

$$\phi(a) = \sum_{(m_1,\dots,m_n)\in\mathbb{N}_0^n} \partial_{x_1}^{(m_1)}\partial_{x_2}^{(m_2)} \cdot \ldots \cdot \partial_{x_n}^{(m_n)}(a) T_1^{m_1} \cdot \ldots \cdot T_n^{m_n}.$$

Let $\mathfrak{p}$ be a prime ideal of A, and consider the composition

$$\bar{\phi} : A \xrightarrow{\phi} A[\![T_1, \dots, T_n]\!] \twoheadrightarrow (A/\mathfrak{p})[\![T_1, \dots, T_n]\!].$$

The ring on the right is an integral domain, so if $\bar{\phi}$ is injective we are done. But the kernel $I := \ker(\bar{\phi})$ is an ideal stable under $\mathscr{D}_K^{(\infty)}$: $a \in I$ if and only if $\partial_{x_1}^{(m_1)} \cdot \ldots \cdot \partial_{x_n}^{(m_n)}(a) \in \mathfrak{p}$ for all $m_1, \dots, m_n \geq 0$. If $a \in I$, then $\partial_{x_1}^{(m_1)} \cdot \ldots \cdot \partial_{x_n}^{(m_n)}(\partial_{x_i}^{(m)}(a)) \in \mathfrak{p}$ for all $m, m_1, \dots, m_n$, because of the composition formulas from Lemma 1.1.2, so $\partial_{x_i}^{(m)}(a) \in I$. Thus $I = 0$ by the definition of a Picard-Vessiot ring, and we are done. ■

1.3.3 Proposition. *Let E be a stratified bundle on K, and $\omega_0 : \langle E \rangle_\otimes \to \mathrm{Vectf}_k$ a fiber functor. Then the affine k-group scheme $G_{\omega_0} := \pi_1(\langle E \rangle_\otimes, \omega_0)$ is smooth.*

If A_{ω_0} is the Picard-Vessiot ring associated with ω_0 (Proposition 1.2.6), and $G_{A_{\omega_0}}$ the differential galois group of E with respect to A_{ω_0}, then there is a canonical k-isomorphism $G_{\omega_0} \cong G_{A_{\omega_0}}$. □

PROOF. By Proposition 1.2.6, we know that the G_{ω_0}-torsor $P_{\omega_0} := \underline{\mathrm{Isom}}_K^\otimes(\omega_0 \otimes_k K, \rho_{\mathrm{forget}})$ on K is represented by a Picard-Vessiot ring A_{ω_0}. By Lemma 1.3.2, P_{ω_0} is reduced, so G_{ω_0} is reduced, hence smooth, because G_{ω_0} is of finite type over k. We also know that there is an isomorphism of $\otimes$-functors $\omega_0 = (- \otimes_K A_{\omega_0})^\nabla$.

The differential Galois group with respect to A_{ω_0} is $G_{A_{\omega_0}} = \mathrm{Aut}_{\mathscr{D}_K^{(\infty)}}(A_{\omega_0})$, and thus there is a canonical inclusion $j : G_{A_{\omega_0}} \hookrightarrow G_{\omega_0}(A_{\omega_0})$. We prove that the image of j coincides with $G_{\omega_0}(k) \subseteq G_{\omega_0}(A_{\omega_0})$. This would finish the proof, since G_{ω_0} and $G_{A_{\omega_0}}$ are both smooth subschemes of $\mathrm{GL}(\omega_0(E)) = \mathrm{GL}((E \otimes A_{\omega_0})^\nabla)$ and k is algebraically closed.

Let $e_1, \dots, e_d$ be a horizontal basis of $E \otimes_K A_{\omega_0}$. We need to show that an A_{ω_0}-linear automorphism ϕ of $E \otimes_K A_{\omega_0}$ is a $\mathscr{D}_K^{(\infty)}$-morphism if and only if the matrix of ϕ with respect to $e_1, \dots, e_d$ has entries in k. Clearly, such a matrix gives a $\mathscr{D}_K^{(\infty)}$-morphism. Conversely, if $\phi(e_r) = \sum a_{sr} e_s$, then for all $i = 1, \dots, n$, and $m \geq 0$:

$$0 = \sum_{s=1}^{d} \partial_{x_i}^{(m)}(a_{sr}) e_s,$$

so $\partial_{x_i}^{(m)}(a_{sr}) = 0$, since A_{ω_0} is an integral domain. But $A_{\omega_0}^\nabla = k$, because A_{ω_0} corresponds to the right regular representation of G_{ω_0}, so the maximal trivial subobject of A_{ω_0} is K, and $K^\nabla = k$. ∎

1.3.4 Proposition. *Let E be a stratified bundle on K, and $\omega_0 : \langle E \rangle_\otimes \to \mathrm{Vectf}_k$ a fiber functor. Then $\pi_1(\langle E \rangle_\otimes, \omega_0)$ is a closed reduced subgroup of a successive extension of $\mathbb{G}_m$'s and $\mathbb{G}_a$'s.* □

PROOF. This is [MvdP03, Cor. 6.4] if $n = 1$. The same proof works: Let $\dim_K E = d$. By Corollary 1.1.14 E is an increasing union $E_1 \subseteq E_2 \subseteq \ldots \subseteq E_d = E$ of substratified bundles, such that $\dim_K E_i = i$. If A_{ω_0} is the Picard-Vessiot ring associated with ω_0, we get an increasing sequence of k-vector spaces

$$(E_1 \otimes_K A_{\omega_0})^\nabla \subseteq (E_2 \otimes_K A_{\omega_0})^\nabla \subseteq \ldots \subseteq (E_d \otimes A_{\omega_0})^\nabla$$

and this flag is fixed by the action of $G_{\omega_0} \subseteq \mathrm{GL}_k(\omega_0(E))$. ∎

1.4 Examples

By Corollary 1.1.14 we know that every stratified bundle E on K is a successive extension of rank 1 stratified bundles, and by Proposition 1.1.12 we know that E is regular singular, if and only if it is a direct sum of rank 1 objects. Hence, for the study of regular singular bundles on $K = k(\!(x_1, \dots, x_n)\!)$, the most important examples to study are rank 1 stratified bundles. In general, the next most important objects to understand are rank 2 stratified bundles which are not regular singular:

1.4.1 Proposition. *Let E be a stratified bundle which is not regular singular. Then $\langle E \rangle_\otimes$ contains a rank 2 stratified bundle which is not regular singular, i.e. a nonsplit extension of two rank 1 stratified bundles.* □

PROOF. Let $E' \in \langle E \rangle_\otimes$ be a stratified bundle with minimal rank d among the objects of $\langle E \rangle_\otimes$ which are not regular singular. Since every rank 1 object is regular singular by Proposition 1.1.10, E' has rank ≥ 2, and by Corollary 1.1.14 there exists a rank 1 sub-stratified bundle $L \subseteq E'$. Then then E'/L is regular singular by the minimality of rank E', so $E'/L = \bigoplus_{i=1}^{d-1} L_i$ with $\operatorname{rank} L_i = 1$ by Proposition 1.1.12. Since $\mathrm{Ext}^1_{\mathrm{Strat}(K)}(L, \bigoplus_{i=1}^{d-1} L_i) = \bigoplus_{i=1}^{d-1} \mathrm{Ext}^1_{\mathrm{Strat}(K)}(L, L_i)$,

E' is a direct sum of extensions of rank 1 stratified bundles. Since E' is not regular singular and has minimal rank with this condition, it follows that $d = 2$ and E' is nontrivial extension of L_1 by L. ∎

Recall that we denote by $\mathcal{O}_K(\alpha_1, \dots, \alpha_n)$ the rank 1 stratified bundle with basis e such that $\delta_{x_i}^{(m)}(e) = \binom{\alpha_i}{m}e$ for $\alpha_i \in \mathbb{Z}_p$. For simplicity, we write $\mathcal{O}_K := \mathcal{O}_K(0, \dots, 0)$.

1.4.2 Proposition. *Let E be a stratified bundle on K.*

(a) *If $E = \bigoplus_{j=1}^d \mathcal{O}_K(\alpha_{1,j}, \dots, \alpha_{n,j})$ then a Picard-Vessiot ring for E is given by*

$$K[T_1^{\pm 1}, \dots, T_d^{\pm 1}]/I$$

where

$$I = \left(T_1^{m_1} \cdots T_d^{m_d} - x_1^{r_1} \cdots x_n^{r_n} \,\middle|\, (r_i)_{1 \le i \le n} = \left(\sum_{j=1}^d m_j \alpha_{i,j} \right)_{1 \le i \le n} \in \mathbb{Z}^n \right)$$

and $\delta_{x_i}^{(m)}(T_r) = \binom{\alpha_{i,r}}{m} T_r$, $r = 1, \dots, d$.

(b) *Accordingly, $\pi_1(\langle E \rangle_\otimes, \omega_0)(k)$ is isomorphic to the subgroup of $(k^\times)^d$ given by*

$$\left\{ (\xi_1, \dots, \xi_d) \in (k^\times)^n \,\middle|\, \prod_{j=1}^d \xi_j^{m_j} = 1, \text{ if } \left(\sum_{j=1}^d m_j \alpha_{i,j} \right)_{1 \le i \le n} \in \mathbb{Z}^n \right\}$$ □

PROOF. (a) We follow Example 1.2.5. Everything follows easily, once one suggestively considers $\mathcal{O}_K(\alpha_{1,j}, \dots, \alpha_{n,j})$ as the rank 1 module with basis $x_1^{\alpha_{1,j}} \cdot \dots \cdot x_n^{\alpha_{n,j}}$. Of course, this does not make sense literally, but it shows what is going on.

To the gritty details: Let $e_1, \dots, e_d$ be a basis of E, such that $\delta_{x_i}^{(m)}(e_j) = \binom{\alpha_{i,j}}{m} e_j$. It is straight forward to check that $T_1^{-1} \cdot \dots \cdot T_d^{-1} e_j$ is horizontal for $j = 1, \dots, d$, so E is trivialized on the differential K-algebra $K[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$, given by $\delta_{x_i}^m(T_r) = \binom{\alpha_{i,r}}{m} T_r$.

It remains to show that the ideal I from the claim is a differential ideal, and maximal amongst the differential ideals: Writing $T^{\mathbf{m}} := T_1^{m_1} \cdot \dots \cdot T_d^{m_d}$ and $x^{\mathbf{r}} := x_1^{r_1} \cdot \dots \cdot x_n^{r_n}$, we get

$$\begin{aligned} \delta_{x_i}^{(m)}(T^{\mathbf{m}} - x^{\mathbf{r}}) &= \binom{\sum_{j=1}^d m_j \alpha_{i,j}}{m} T^{\mathbf{m}} - \binom{r_i}{m} x^{\mathbf{r}} \\ &= \binom{r_i}{m} (T^{\mathbf{m}} - x^{\mathbf{r}}) \end{aligned}$$

if $T^{\mathbf{m}} - x^{\mathbf{r}} \in I$, so I is indeed a differential ideal.

To prove that I is maximal, one distinguishes two cases: $I = 0$ and $I \neq 0$, and then the computation is very similar to [vdP99, Ex. 3.5].

(b) This can be read off directly from the description of I. ∎

1.4.3 Corollary. *A stratified bundle E on K is regular singular if and only if $\pi_1(\langle E\rangle_{\otimes}, \omega_0)$ is diagonizable for some (or equivalently any) neutral fiber functor ω_0.* □

PROOF. By Corollary 1.1.13, E is regular singular if and only if it is a direct sum of rank 1 subbundles. Proposition 1.4.2 shows that the differential galois group of a direct sum of rank 1 bundles is diagonizable.

Conversely, by Proposition 1.4.1, if E is not regular singular, then the differential galois group $\pi_1(\langle E\rangle_{\otimes}, \omega_0)$ has a quotient which is a closed reduced subgroup of $\mathbb{G}_a$, so $\pi_1(\langle E\rangle_{\otimes}, \omega_0)$ cannot be diagonizable. ∎

1.4.4 Example. Finally some concrete examples; the case $n = 1$ is due to [MvdP03], and most of the arguments carry through.

(a) Let $\alpha_1, \dots, \alpha_n \in \mathbb{Z}_p$ and $\alpha_1 \in \mathbb{Z}$. Then

$$\mathscr{O}_K(\alpha_1, \dots, \alpha_n) \cong \mathscr{O}_K(0, \alpha_2, \dots, \alpha_n)$$

as stratified bundles. Indeed, the K-morphism $1 \mapsto x_1^{\alpha}$ defines an isomorphism of stratified bundles

$$\mathscr{O}_K(\alpha_1, \dots, \alpha_n) \to \mathscr{O}_K(0, \alpha_2, \dots, \alpha_n).$$

Thus, if $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$, then $\mathscr{O}_K(\alpha_1, \dots, \alpha_n) \cong \mathscr{O}_K$, which can also be read off from the description of the monodromy group in Proposition 1.4.2.

(b) Now assume that $\alpha_1, \dots, \alpha_n \in \mathbb{Z}_p$, and $\alpha_1 \notin \mathbb{Q} \cap \mathbb{Z}_p$. Then the differential galois group G of $\mathscr{O}_K(\alpha_1, \dots, \alpha_n)$ is isomorphic to $\mathbb{G}_m$ since by Proposition 1.4.2,

$$G(k) = \left\{\xi \in k^{\times}\, | \xi^{m_i} = 1, \text{ if } m_i\alpha_i \in \mathbb{Z}, i = 1, \dots, n\right\}$$

(c) Contrastingly, let $\alpha_1, \dots, \alpha_n \in \mathbb{Q} \cap \mathbb{Z}_p$, and $\alpha_i = r_i/s_i$ with $r_i, s_i \in \mathbb{Z}$, $(r_i, s_i) = 1$, $(s_i, p) = 1$ and $s_i > 0$. If $s \geq 0$ is the smallest common multiple of $s_1, \dots, s_n$, then a Picard-Vessiot extension for $\mathscr{O}_K(\alpha_1, \dots, \alpha_n)$ is isomorphic to the Kummer extension of K given by adjoining an s-th root of $x_1 \cdots x_n$, and the differential Galois group of $\mathscr{O}_K(\alpha_1, \dots, \alpha_n)$ is $\boldsymbol{\mu}_s$.

(d) Conversely, consider the field extension $E := K(x_1^{1/\ell_1}, \dots, x_n^{1/\ell_n})$. For simplicity, assume that $\ell_1, \dots, \ell_n$ are pairwise distinct primes which are also all different from p. The $\mathscr{D}_K^{(\infty)}$-action extends uniquely to E, and E is a stratified bundle of dimension $\prod_{i=1}^n \ell_i$ with differential galois group $\prod_{i=1}^n \boldsymbol{\mu}_{\ell_i}$. More precisely:

$$E \cong \bigoplus_{0 \leq r_i \leq \ell_i - 1} \mathscr{O}_K(r_1/\ell_1, \dots, r_n/\ell_n).$$

Indeed, $K(x_1^{1/\ell_1})$ is the stratified bundle

$$\bigoplus_{0 \leq r_1 < \ell_1} \mathscr{O}_K(r_1/\ell_1, 0, \dots, 0);$$

$K(x_1^{1/\ell_1}, x_2^{1/\ell_2})$ is the stratified bundle

$$\bigoplus_{0\leq r_2<\ell_2-1} \mathcal{O}_{K(x_1^{1/\ell_1})}(0, r_2/\ell_2, 0\ldots, 0);$$

and so forth.

(e) More generally, let L/K be any finite separable extension. Then L is a stratified bundle of rank $[L:K]$: We can uniquely extend the $\mathscr{D}_K^{(\infty)}$-action of K to L because of the separability of L/K. If L is also galois with group G over K, then $G = \mathrm{Gal}(L/K) \cong \pi_1(\langle L\rangle_\otimes, \omega_0)$, for any fiber functor ω_0. In fact, L is the Picard-Vessiot ring for itself, and any automorphism of the extension L/K is a $\mathscr{D}_K^{(\infty)}$-automorphism. For a global analog with proof, see Corollary 4.1.5.

(f) As another particular case of the previous example, consider the Artin-Schreier extension of K given by $L := K[u]/(u^p - u - x_1^{-1})$. This is a separable field extension, and the $\mathscr{D}_K^{(\infty)}$-action extends unquely to E: For all $m > 0$, we have

$$u^{p^{m+1}} - u = \sum_{i=1}^{m} x_1^{-p^i} \tag{1.7}$$

as can be easily checked via induction. This means we can define

$$\delta_{x_j}^{(p^m)}(u) = -\sum_{i=1}^{m} \delta_{x_j}^{(p^m)}\left(x_1^{-p^i}\right) = \begin{cases} 0 & j>1 \\ \sum_{i=1}^{m} x_1^{-p^i} & \text{else} \end{cases},$$

because by Lemma 1.1.9

$$\binom{-p^i}{p^m} \equiv \binom{\sum_{j\geq i}(p-1)p^j}{p^m} \equiv -1 \mod p$$

if $i \leq m$.

Consider the K-subspace $E \subseteq L$ generated by 1 and u. Then E is stable by the $\mathscr{D}_K^{(\infty)}$-action, and $\mathcal{O}_K = 1\cdot K \subseteq E$ is a substratified bundle. Let E' be the quotient $E/\mathcal{O}_K$. We get a short exact sequence

$$0 \to \mathcal{O}_K \to E \to E' \to 0$$

As a K-vector space E' is generated by u, and the $\mathscr{D}_K^{(\infty)}$-structure is given by $\delta_{x_i}^{(p^m)}(u) = 0$ in E'. In other words: $E' \cong \mathcal{O}_K$. We see that E' is a nontrivial extension of $\mathcal{O}_K$ by $\mathcal{O}_K$, hence cannot be regular singular.

We compute its Picard-Vessiot extension and its differential galois group as in Example 1.2.5: An element $y = y_1 + y_2u \in E$ lies in E^∇ if and only if

$$\begin{pmatrix} \delta_1^{(p^m)}(y_1) \\ \delta_1^{(p^m)}(y_2) \end{pmatrix} = \begin{pmatrix} 0 & -\sum_{i=1}^{m} x_1^{-p^i} \\ 0 & 0 \end{pmatrix} \cdot y$$

and $\delta_j^{(p^m)}(y_1) = \delta_j^{(p^m)}(y_2) = 0$ for $j > 1$. Consider the ring $R_0 := K[T_{11}, T_{12}, T_{21}, T_{22}, 1/\det(T_{ij})]$ with the $\mathscr{D}_K^{(\infty)}$-action given by

$$\begin{pmatrix} \delta_{x_1}^{(p^m)}(T_{11}) & \delta_{x_1}^{(p^m)}(T_{12}) \\ \delta_{x_1}^{(p^m)}(T_{21}) & \delta_{x_1}^{(p^m)}(T_{22}) \end{pmatrix} = \begin{pmatrix} 0 & -\sum_{i=1}^m x_1^{-p^i} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

and $\delta_j^{(p^m)}(T_{ij}) = 0$ for $j > 1$. Then $R_0 \otimes E$ is trivial with horizontal basis $T_{11} + uT_{21}$, $T_{12} + uT_{22}$, and we need to find an ideal in R_0 maximal amongst those which are $\mathscr{D}_K^{(\infty)}$-stable. For example, we can take

$$I := (T_{21}, T_{22} - 1, T_{11} - 1, T_{12}^p - T_{12} + x_1^{-1}).$$

It is easily checked that this is a $\mathscr{D}_K^{(\infty)}$-stable, maximal ideal of R_0: Only the generator $T_{12}^p - T_{12} + x^{-1}$ requires a little calculation: I claim that for all $r, m \geq 0$:

$$\delta_{x_1}^{(m)}(T_{11}^r) = T_{22}^r \delta_{x_1}^{(m)}\left(\left(\sum_{i=0}^s u^{-p^i}\right)^r\right)$$

for $p^s \geq m$. Then

$$\begin{aligned} \delta_{x_1}^{(p^m)}(T_{11}^p - T_{11} + x_1^{-1}) &= -T_{22}^p \cdot \sum_{i=1}^m x_1^{-p^i} + T_{22} \sum_{i=0}^m x_1^{-p^i} - x_1^{-1} \\ &= -(T_{22}^p - T_{22}) \sum_{i=1}^m x_1^{-p^i} + (T_{22} - 1)x_1^{-1} \end{aligned}$$

which lies in I, as $T_{22} - 1$ is a factor of $T_{22}^p - T_{22}$.

It follows that R_0/I is isomorphic as a K-algebra to the Artin-Schreier extension L we started with, and that the differential galois group is isomorphic to the galois group $\mathbb{G}_m(\mathbb{F}_p) = \mathbb{Z}/p\mathbb{Z}$ of L/K.

(g) Now, more generally, we consider an extension

$$0 \to \mathcal{O}_K \to E \to \mathcal{O}_K \to 0.$$

It is clear from tannaka theory, that $\pi_1(\langle E \rangle_\otimes, \omega_0)$ is a closed reduced subgroup of $\mathbb{G}_a$, for any fiber functor ω_0. Indeed, every automorphism of ω_0 gives an K-linear automorphism of E fixing the subobject $\mathcal{O}_K$ and the quotient $\mathcal{O}_K$.

Let e_1, e_2 be a basis of E, such that $\partial_{x_i}^{(m)}(e_1) = 0$ and $\partial_{x_i}^{(m)}(e_2) = a_i^{(m)} e_1$ for $a_i^{(m)} \in K$, $i = 1, \ldots, n$, $m \geq 0$. Of course the $a_i^{(m)}$ satisfy certain relations, coming from the relations of differential operators in $\mathscr{D}_K^{(\infty)}$. In particular, for every $i = 1, \ldots, n$, the sequence $(a_i^{(m)})_{m \geq 0}$ is determined by the subsequence $(a_i^{(p^m)})_{m \geq 0}$, and since $\partial_{x_j}^{(m)} \partial_{x_i}^{(m')} = \partial_{x_i}^{(m')} \partial_{x_i}^{(m)}$, it follows that

$$\partial_{x_i}^{(m)}(a_j^{(m')}) = \partial_{x_j}^{(m')}(a_i^{(m)}),$$

Let $y = y_1 e_1 + y_2 e_2 \in E$. Then $y \in E^\nabla$ if and only if $\partial_{x_i}^{(m)}(y_2) = 0$ and $\partial_{x_i}^{(m)}(y_1) = -y_2 a_i^{(m)}$. In particular $y_2 \in k$.

At this point, it is convenient to use the description of Proposition 1.1.6 to further analyze the extension E, following [MvdP03]: Let E correspond to $(E_m, \sigma_m)_{m\geq 0}$, and fix bases $e_1^{(m)}, e_2^{(m)}$ of E_m. Then the isomorphism

$$\sigma_0 \circ \sigma_1 \circ \ldots \circ \sigma_m : K^{p^m} \otimes_{K^{p^m}} E_{m+1} \xrightarrow{\cong} E_0$$

is given by a matrix

$$\begin{pmatrix} 1 & b_m \\ 0 & 1 \end{pmatrix}, \; b_m \in K.$$

We may change the basis of E_{m+1}, but any such base change has the form

$$\begin{pmatrix} 1 & c_m \\ 0 & 1 \end{pmatrix}, \; c_m \in K^{p^{n+1}}$$

so b_m is well defined modulo $K^{p^{m+1}}$. Moreover, $b_n \equiv b_{n+1} \mod K^{p^n}$. Thus E corresponds to an element of $\varprojlim_{m\geq 0} K/K^{p^{m+1}}$. Moreover, E is split if and only if it corresponds to an element in

$$\operatorname{im}(K \to \varprojlim_{m\geq 0} K/K^{p^{m+1}}),$$

so we get a bijection

$$\operatorname{Ext}^1_{\operatorname{Strat}(K)}(\mathcal{O}_K, \mathcal{O}_K) \cong \operatorname{coker}\left(K \to \varprojlim_{m\geq 0} K/K^{p^{m+1}}\right).$$

It is not hard so see that this is actually an isomorphism of groups.

We can recover the $a_i^{(p^m)}$ from above:

$$\partial_{x_i}^{(p^m)}(e_2^{(0)}) = \partial_{x_i}^{(p^m)}(e_1^{(m)} - b_m e_2^{(m)}) = -\partial_{x_i}^{(p^m)}(b_m).$$

(h) Finally, in contrast to the situation in characteristic 0 ([Kat87, Lemma 2.3.5]), if L_1, L_2 are two rank 1 stratified bundles on K, then

$$\operatorname{Ext}^1_{\operatorname{Strat}(K)}(L_1, L_2) \neq 0$$

does not imply that $L_1 \cong L_2$. This follows from [MvdP03, Thm. 6.6]. For example, *loc. cit.* implies that there is a rank 2 stratified bundle on $k((t))$, the differential galois group of which is the group of matrices of the form

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix},$$

i.e. the group semi-direct product of $\mathbb{G}_a \rtimes \mathbb{G}_m$ with $\mathbb{G}_m$ acting on $\mathbb{G}_a$ via $\mu \cdot a = \mu a + 1$. $\square$

Chapter 2

Logarithmic differential operators, connections and stratifications

In this chapter we construct the sheaf of logarithmic differential operators with respect to a strict normal crossings divisor, and define logarithmic connections, logarithmic n-connections, logarithmic stratifications and their exponents. The construction is parallel to the classical ("non-logarithmic") construction: From a k-scheme X with strict normal crossings divisor D (or more generally, a morphism of fine log-schemes), we first construct suitable sheaves of "logarithmic principal parts" $\mathcal{P}^n_{X/k}(\log D)$ as structure sheaves of thickenings of a suitable diagonal, and show that they satisfy the necessary relations such that $\mathscr{D}_{X/k}(\log D) := \varinjlim \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/k}(\log D), \mathcal{O}_X)$ is a sheaf of rings; the sheaf of "logarithmic differential operators". Due to the general theory of "groupoïds formels" (which we will call "formals groupoids", see Appendix D) developed by P. Berthelot in [Ber74], the theory of n-connections, stratifications and differential operators follows, and takes a similar shape to its the non-logarithmic counterpart. A notable difference is that on a smooth k-variety X, an $\mathcal{O}_X$-coherent $\mathscr{D}_{X/k}(\log D)$-module is not necessarily locally free (see e.g. the example after [AB01, Def. 4.4]). The construction of exponents is due to N. Katz and D. Gieseker, [Gie75].

The content of this chapter is probably well-known to experts, but the author does not know of a complete reference. We refer however to [Mon02] for the construction of log-differential operators with divided powers for a morphism of fine log-schemes.

2.1 A prelude on logarithmic geometry

2.1.1 Generalities about log-schemes

While the explicit use of logarithmic schemes is not needed in later chapters, they do provide a natural general context for Chapter 2. Nevertheless, the constructions from this section will be made more explicit in Section 2.2, so the reader not familiar with logarithmic schemes can skip this Section 2.1 on a first reading.

The results of this section are mostly contained in [Mon02] and [AB01, Appendix 1.B]. Hence we limit ourselves to a brief summary. We do however give a full construction of the sheaves of principal parts, in part because the important Remark 2.1.13, (b) is usually not stated explicitly. Lecture notes of M. Cailotto on logarithmic schemes have been helpful. For details on log-schemes, see e.g. [Kat89].

Recall the following definitions:

2.1.1 Definition. By a *monoid* P we mean a set P equipped with a binary operation, which is associative, commutative, and has a neutral element.

- $P^\times$ denotes the submonoid of invertible elements of P. It is an abelian group.
- P^{gp} denotes the abelian group obtained by formally inverting the elements of P. There is a canonical morphism of groupoids $P \to P^{\mathrm{gp}}$.
- P is called *integral* if $P \to P^{\mathrm{gp}}$ is injective.
- $P^{\mathrm{int}} := \mathrm{im}(P \to P^{\mathrm{gp}})$ is the *integralization of* P. It is an integral monoid, and for every integral monoid Q, every map $P \to Q$, factors uniquely through $P \to P^{\mathrm{int}}$.
- if P is integral, then the *saturation* P^{sat} *of* P is the monoid given by
$$P^{\mathrm{sat}} = \{a \in P^{gp} | \exists n \geq 0 : a^n \in P\} .$$
- P is called *saturated* if P is integral, and if $P = P^{\mathrm{sat}}$. For every saturated monoid Q, every map $P \to Q$ factors uniquely through $P \to P^{\mathrm{sat}}$. □

2.1.2 Definition. Let X be a scheme. A *pre*-log-*structure* (M_X, α_X) is a sheaf of monoids M_X on the étale site $X_{\text{ét}}$, and a morphism $\alpha_X : M_X \to (\mathcal{O}_X, \cdot)$ of monoid sheaves. The pair (M_X, α_X) is called log-*structure* if $\alpha_X^{-1}\mathcal{O}_X^\times \xrightarrow{\cong} \mathcal{O}_X^\times$ is an isomorphism. The triple (X, M_X, α_X) is called log-*scheme*. If (Y, M_Y, α_Y) and (X, M_X, α_X) are two log-schemes, then a morphism of log-schemes $(f, g) : (Y, M_Y, \alpha_Y) \to (X, M_X, \alpha_X)$ is a morphism of schemes $f : Y \to X$, together with a morphism of sheaves of monoids $g : f^{-1}(M_X) \to M_Y$, such that the diagram

$$\begin{tikzcd}
f^{-1}M_X \arrow[r, "g"] \arrow[d, "f^{-1}\alpha_X"'] & M_X \arrow[d] \\
f^{-1}\mathcal{O}_X \arrow[r] & \mathcal{O}_Y
\end{tikzcd}$$

commutes.

There is a functorial way to associate with a pre-log-structure a log-structure, and this construction is adjoint to the forgetful functor from the category of log-structure to the category of pre-log-structures. If $f : Y \to X$ is a morphism of schemes, and M_X a log-structure on X, then we write f^*M_X for the log-structure on Y, associated with the pre-log-structure $f^{-1}M_X \to f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$. We call it the *pull-back of* M_X.

If $f : (X, M_X) \to (Y, M_Y)$ is a morphism of log-schemes, then f induces a functorial morphism $f^* M_Y \to M_X$. □

2.1.3 Remark.

- We will usually just write (X, M_X) for a log-scheme and f for a morphism of log-schemes, dropping the morphisms α_X and g from the notation, to increase legibility.
- One can also define (pre)-log-structures with respect to the Zariski topology. In all of our applications, it will not matter which topology we choose, see Section 2.1.4. □

2.1.4 Definition. Let $(X, M_X), (Y, M_Y)$ be log-schemes, and $f : (X, M_X) \to (Y, M_Y)$ a morphism of log-schemes.

- (X, M_X) is called *integral* if M_X is a sheaf of integral monoids.
- (X, M_X) is called *coherent* if étale locally on X, there exists a finitely generated monoid P and a morphism $\beta : P \to \mathcal{O}_X$, such that log-structure associated with β is isomorphic to M_X.
- (X, M_X) is called *fine* if it is integral and coherent.
- (X, M_X) is called *saturated* if (X, M_X) is integral and if for every geometric point $\bar{x}$ of X, the monoid $(M_X)_{\bar{x}}$ is saturated.
- f is called *closed immersion*, if the underlying morphism of schemes is a closed immersion, and if the induced map $f^* M_Y \to M_X$ is surjective.
- f is called *exact*, if for every geometric point $\bar{x} \to X$ the morphism $f_{\bar{x}} : (f^{-1} M_Y)_{\bar{x}} \to M_{X,\bar{x}}$ is an exact morphism of monoids, which means that
$$(f^{-1} M_Y)_{\bar{x}} = (f_{\bar{x}}^{gp})^{-1}(M_{X,\bar{x}}) \subseteq ((f^{-1} M_Y)_{\bar{x}})^{gp}.$$
- f is called *strict*, if the induced morphism $f^* M_Y \to M_X$ is an isomorphism. □

2.1.5 Definition. Let P be a monoid and $\mathbb{Z}[P]$ the monoid ring. Then the morphism of monoids $e_P : P \to \mathbb{Z}[P]$ defines a pre-log-structure on $\operatorname{Spec} \mathbb{Z}[P]$, and from now on we consider $\operatorname{Spec} \mathbb{Z}[P]$ as a log-scheme with the log-structure associated with e_P.

If (X, M_X) is a log-scheme, then a *chart for* (X, M_X) is a strict morphism of log-schemes $(X, M_X) \to \operatorname{Spec} \mathbb{Z}[P]$. The log-scheme (X, M_X) is coherent if and only if (X, M_X) étale locally admits a chart by a finitely generated monoid.

If $f : (X, M_X) \to (Y, M_Y)$ is a morphism of log-schemes, then a *chart for* f is a triple (a, b, c), where $a : (X, M_X) \to \operatorname{Spec} \mathbb{Z}[P]$, $: (Y, M_Y) \to \operatorname{Spec} \mathbb{Z}[Q]$ are charts for (X, M_X) and (Y, M_Y), and $c : Q \to P$ a morphism of monoids, such that the obvious diagram commutes. If (X, M_X), (Y, M_Y) are fine log-schemes, then a chart for f always exits étale locally [Kat89, 2.10]. □

2.1.6 Remark.

- Note that for an exact morphism $f : (X, M_X) \to (Y, M_Y)$ the morphism $f^{-1}M_Y \to M_X$ is injective. Thus a closed immersion of log-schemes is exact if and only if $f^{-1}M_X \to M_Y$ is an isomorphism of log-structures. In other words: *A closed immersion is strict if and only if it is exact.*

- For (X, M_X) a coherent log-scheme, there is a fine log-scheme $(X, M_X)^{\mathrm{int}}$ and a closed immersion $i_X^{\mathrm{int}} : (X^{\mathrm{int}}, M_X^{\mathrm{int}}) \hookrightarrow (X, M_X)$, such that for every integral log-scheme (Y, M_Y), every morphism $(Y, M_Y) \to (X, M_X)$ factors uniquely through i_X^{int}. See [Kat89, Prop (2.7)]. Indeed, if $\operatorname{Spec} \mathbb{Z}[P] \to (X, M_X)$ is a chart, then $X^{\mathrm{int}} := X \times_{\mathbb{Z}[P]} \mathbb{Z}[P^{\mathrm{int}}]$, with the pull-back log-structure.

- For every integral log-scheme (X, M_X), there is a saturated log-scheme $(X, M_X)^{\mathrm{sat}}$ and a morphism $i_X^{\mathrm{sat}} : (X^{\mathrm{sat}}, M_X^{\mathrm{sat}}) \hookrightarrow (X, M_X)$ with $X^{\mathrm{sat}} \to X$ finite, such that for every saturated log-scheme (Y, M_Y), every morphism $(Y, M_Y) \to (X, M_X)$ factors uniquely through i_X^{sat}. Indeed, if $\operatorname{Spec} \mathbb{Z}[P] \to X$ is a chart, then $X^{\mathrm{sat}} := X \times_{\mathbb{Z}[P]} \mathbb{Z}[P^{\mathrm{sat}}]$ with the pull-back log-structure. (Note that if P is finitely generated, then P^{sat} is fine and saturated.) □

2.1.7 Proposition (follows from [Kat89, Prop. 4.10]). *If $i : (X, M_X) \to (Y, M_Y)$ is a closed immersion of fine* log-*schemes, then étale locally i factors as*

$$(X, M_X) \xrightarrow{i'} (Z, M_Z) \xrightarrow{f} (Y, M_Y) \tag{2.1}$$

with i' an exact (and thus strict) closed immersion, f log-*étale and M_Z fine.* □

PROOF. We recall the construction of [Kat89, Prop. 4.10], because we will need it later on: We may assume that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and that there are finitely generated monoids P, Q, and charts $(X, M_X) \to \operatorname{Spec} \mathbb{Z}[P]$, $(Y, M_Y) \to \operatorname{Spec} \mathbb{Z}[Q]$, where the rings $\mathbb{Z}[P]$ and $\mathbb{Z}[Q]$ are endowed with their canonical log-structures. Moreover, we may also assume that there is a morphism of monoids $h : Q \to P$, such that the diagram

$$\begin{tikzcd}
(X, M_X) \arrow[r] \arrow[d, "i"'] & \operatorname{Spec} \mathbb{Z}[P] \arrow[d, "\operatorname{Spec} \mathbb{Z}[h]"] \\
(Y, M_Y) \arrow[r] & \operatorname{Spec} \mathbb{Z}[Q]
\end{tikzcd}$$

commutes, see [Kat89, (2.9)]. Define $Q' := (h^{gp})^{-1}(P)$, $Z := Y \times_{\mathbb{Z}[Q]} \mathbb{Z}[Q']$, and M_Z as the pull-back of the log-structure of $\mathbb{Z}[Q']$. M_Z is a fine log-structure. As h induces an exact morphism of monoids $Q' \to P$, we get an exact morphism $i' : (X, M_X) \to (Z, M_Z)$, such that post-composition with the projection f to (Y, M_Y) is equal to i. This gives a factorization (2.1). The projection f is log-étale by [Kat89, (3.5.2)]. It remains to show that i' is an exact closed immersion: Exactness follows from the fact that the map $Q' \to P$ is exact, and it is a closed immersion, because $X \to Z$ is a section of the separated map $X \times_Y Z \to X$. ■

2.1.8 Remark. (a) Let (Y, M_Y) be a coherent log-scheme, not necessarily fine, and $i_Y^{\mathrm{int}} : (Y^{\mathrm{int}}, M_Y^{\mathrm{int}}) \hookrightarrow (Y, M_Y)$, the universal closed immersion (see

Remark 2.1.6). If (X, M_X) is fine and $j : (X, M_X) \hookrightarrow (Y, M_Y)$ a closed immersion, then i factors uniquely as

$$(X, M_X) \xhookrightarrow{j^{\mathrm{int}}} (Y^{\mathrm{int}}, M_Y^{\mathrm{int}}) \xhookrightarrow{i_Y^{\mathrm{int}}} (Y, M_Y) .$$

We can compute the factorization (2.1) given in the proof of the proposition for j^{int} in terms of (Y, M_Y):

Assume we have global charts

$$X \to \operatorname{Spec} \mathbb{Z}[P],\ Y \to \operatorname{Spec} \mathbb{Z}[Q],\ h : Q \to P$$

such that we get a commutative diagram

$$\begin{tikzcd}
(X, M_X) \arrow[r] \arrow[d] \arrow[dd, bend right, "j"'] & \operatorname{Spec} \mathbb{Z}[P] \arrow[d, "h^{\mathrm{int}}"] \arrow[dd, bend left, "\operatorname{Spec} \mathbb{Z}[h]"] \\
(Y^{\mathrm{int}}, M_Y^{\mathrm{int}}) \arrow[r] \arrow[d, "i_Y^{\mathrm{int}}"] & \operatorname{Spec} \mathbb{Z}[Q^{\mathrm{int}}] \arrow[d] \\
(Y, M_Y) \arrow[r] & \operatorname{Spec} \mathbb{Z}[Q]
\end{tikzcd} ,$$

with exact rows. Recall that $Q^{\mathrm{int}} = \mathrm{im}(Q \to Q^{gp})$. As in the proof of the proposition, let $Q' := ((h^{\mathrm{int}})^{gp})^{-1}(P) = (h^{gp})^{-1}(P)$. Defining $Z := Y^{\mathrm{int}} \times_{\operatorname{Spec} \mathbb{Z}[Q^{\mathrm{int}}]} \operatorname{Spec} \mathbb{Z}[Q']$ and M_Z as the pull-back of the log-structure of $\mathbb{Z}[Q']$, we get the factorization (2.1)

$$(X, M_X) \xhookrightarrow[\text{closed}]{\text{exact}} (Z, M_Z) \xrightarrow{\text{log-étale}} (Y^{\mathrm{int}}, M_Y^{\mathrm{int}}) .$$

But clearly $Z \cong Y \times_{\mathbb{Z}[Q]} \mathbb{Z}[Q']$, which describes (Z, M_Z) solely in terms of (Y, M_Y).

(b) If (X, M_X) is saturated and fine, and $j : (X, M_X) \hookrightarrow (Y, M_Y)$ a closed immersion, then the construction from the proof of the proposition gives two factorizations

$$(X, M_X) \xhookrightarrow[\text{closed}]{\text{exact}} (Z_1, M_{Z_1}) \xrightarrow{\text{log-étale}} (Y^{\mathrm{sat}}, M_Y^{\mathrm{sat}}) \longrightarrow (Y^{\mathrm{int}}, M_Y^{\mathrm{int}})$$

and

$$(X, M_X) \xhookrightarrow[\text{closed}]{\text{exact}} (Z_2, M_{Z_2}) \xrightarrow{\text{log-étale}} (Y^{\mathrm{int}}, M_Y^{\mathrm{int}})$$

The same argument as in (a) shows that there is an isomorphism $\phi : (Z_1, M_{Z_1}) \to (Z_2, M_{Z_2})$, such that we get a commutative diagram

$$\begin{tikzcd}
(X, M_X) \arrow[r, hook, "\text{exact}", "\text{closed}"'] \arrow[d, equal] & (Z_1, M_{Z_1}) \arrow[r, "\text{log-étale}"] \arrow[d, "\phi", "\cong"'] & (Y^{\mathrm{sat}}, M_Y^{\mathrm{sat}}) \arrow[r] & (Y^{\mathrm{int}}, M_Y^{\mathrm{int}}) \arrow[d, equal] \\
(X, M_X) \arrow[r, hook, "\text{exact}", "\text{closed}"'] & (Z_2, M_{Z_2}) \arrow[rr, "\text{log-étale}"] & & (Y^{\mathrm{int}}, M_Y^{\mathrm{int}})
\end{tikzcd}$$

See also [AB01, Lemma B.4.2]. □

2.1.2 Logarithmic principal parts

Construction and basic properties

The factorizations constructed in the previous section allow us to define infinitesimal neighborhoods in the category of log-schemes:

2.1.9 Proposition (comp. with [Mon02, Prop. 2.1.1]). *Let the logarithmic scheme (S, N) be fine, and (X, M_X) and (Y, M_Y) fine* log-*schemes over (S, N). Let $i : (X, M_X) \to (Y, M_Y)$ be a closed immersion. Then there exists a fine (S, N)*-log-*scheme $(\Delta^n_X(Y), M_n)$ and a factorization*

$$(X, M_X) \xhookrightarrow{i_n} (\Delta^n_X(Y), M_n) \xrightarrow{f_n} (Y, M_Y)$$

over (S, N), such that i_n is an exact closed immersion, f_n log-*étale, satisfying the following universal property: For any commutative diagram of (S, N)*-log-*schemes*

$$\begin{tikzcd}
(X', M_{X'}) \arrow[rr, hook, "i'"] \arrow[dd] & & (Y', M_{Y'}) \arrow[dl, dotted, "f'"'] \arrow[dd, "f"] \\
& (\Delta^n_X(Y), M_n) \arrow[dr, "f_n"] & \\
(X, M_X) \arrow[ur, "i_n"] \arrow[rr, hook, "i"] & & (Y, M_Y)
\end{tikzcd}$$

where i' is an exact closed immersion of fine log schemes, such that $X' \hookrightarrow Y'$ is defined by a sheaf of ideals $\mathcal{I}'$ with $\mathcal{I}'^{n+1} = 0$, there exists a unique dotted arrow $f' : (Y', M_{Y'}) \to (\Delta^n_X(Y), M_n)$ making the diagram commutative. □

PROOF. The idea is as follows: Factor i locally via Proposition 2.1.7 as

$$(X, M_X) \xrightarrow{i'} (Z, M_Z) \xrightarrow{f} (Y, M_Y) ,$$

take $\Delta^n_X(Y)$ to be the n-th infinitesimal neighborhood of X in Z, and M_n as the pull-back of M_Z. Because of the universal property, these constructions glue the global object $(\Delta^n_X(Y), M_n)$.

To verify the universal property, note that the log-étaleness of the morphism $(Z, M_Z) \to (Y, M_Y)$, implies that the dotted arrow of the following diagram exists:

$$\begin{tikzcd}
(X', M_{X'}) \arrow[r, hook, "\text{exact}"] \arrow[d] & (Y', M_{Y'}) \arrow[ddl, dotted, "u"] \arrow[dd] \\
(X, M_Z) \arrow[d] & \\
(Z, M_Z) \arrow[r, "\text{log-étale}"'] & (Y, M_Y)
\end{tikzcd}$$

As $X' \hookrightarrow Y'$ is defined by an ideal $\mathcal{I}'$ with $\mathcal{I}'^{n+1} = 0$, u induces a unique morphism v of *schemes* $Y' \to \Delta^n_X(Y)$, and as M_n is the pull-back of M_Z, v extends (uniquely) to a morphism of log-schemes $(Y', M_{Y'}) \to (\Delta^n_X(Y), M_n)$. ■

2.1.10 Remark. In particular $(\Delta_X^n(Y), M_n)$ is independent of the chosen local factorizations. □

2.1.11 Definition. With the notation from the proposition, we call the log-scheme $(\Delta_X^n(Y), M_n)$ together with the maps i_n and f_n the *n-th logarithmic infinitesimal neighborhood* of X in Y.

2.1.12 Definition. Let $(X, M_X) \to (S, M_S)$ be a finite type morphism of fine log-schemes, and consider the diagonal map

$$\Delta : (X, M_X) \to (X, M_X) \times_{(S,M_S)} (X, M_X)$$

in the category of log-schemes. The underlying morphism of schemes is the usual diagonal $X \to X \times_S X$ (because we took the product in the category of log-schemes, not fine log-schemes) and it is an immersion; let $U \subseteq X \times_S X$ be an open subset, such that $\Delta(X) \subseteq U$ is a closed subset. Let M_U be the induced log-structure on U. Let $(U^{\mathrm{int}}, M_U^{\mathrm{int}})$ be the integral log-structure associated with U and $U^{\mathrm{int}} \hookrightarrow U$ the canonical closed immersion. Then Δ factors via a unique closed immersion $(X, M_X) \hookrightarrow (U^{\mathrm{int}}, M_U^{\mathrm{int}})$, and, abusing notation, we write (Δ_X^n, M_X^n) for the n-th logarithmic infinitesimal neighborhood of (X, M_X) in $(U^{\mathrm{int}}, M_U^{\mathrm{int}})$. We can consider the structure sheaf of (Δ_X^n, M_X^n) as a sheaf on X, and denote it by $\mathcal{P}^n_{(X,M_X)/(S,M_S)}$ and as an $\mathcal{O}_X$-bimodule via the two projections. We call $\mathcal{P}^n_{(X,M_X)/(S,M_S)}$ the *sheaf of n-th logarithmic principal parts.* □

2.1.13 Remark. (a) It is easy to see that $\mathcal{P}^n_{(X,M_X)/(S,M_S)}$ is independent of the choice of U: If V, U are two an open subsets of $X \times_S X$ such that $\Delta(X) \subseteq V$ is a closed subset, then we get a commutative diagram

$$\begin{tikzcd}
 & (U^{\mathrm{int}}, M_U^{\mathrm{int}}) \\
(X, M_X) \arrow[ur] \arrow[r] \arrow[dr] & ((U \cap V)^{\mathrm{int}}, M_{U\cap V}^{\mathrm{int}}) \arrow[u] \arrow[d] \\
 & (V^{\mathrm{int}}, M_V^{\mathrm{int}})
\end{tikzcd}$$

with open immersions as vertical maps. A factorization of $\Delta_{U\cap V}$ as in Proposition 2.1.7 thus gives factorizations of Δ_U and Δ_V as well, and the infinitesimal neighborhoods are independent of the chosen factorizations, see Remark 2.1.10.

(b) If (S, M_S) is a fine (resp. fine and saturated) log-scheme and (X, M_X) a fine (resp. fine and saturated) (S, M_S)-log-scheme, then the product of (X, M_X) with itself in the category of (S, M_S)-log-schemes is as a log-scheme a priori different from the product in the category of fine (S, M_S)-log-schemes (resp. in the category of fine and saturated (S, M_S)-log-schemes). By Remark 2.1.8, the sheaves of principal parts $\mathcal{P}^n_{(X,M_X)/(S,M_S)}$ do *not* depend on the category in which we consider the diagonal morphism. □

By construction, for every $n \geq 0$, we have exact closed immersions

$$(\Delta_X^n, M_X^n) \hookrightarrow (\Delta_X^{n+1}, M_X^{n+1}),$$

and accordingly a sequence of surjective morphisms

$$u : \mathcal{P}^{n+1}_{(X,M_X)/(S,M_S)} \twoheadrightarrow \mathcal{P}^{n}_{(X,M_X)/(S,M_S)}. \tag{2.2}$$

This makes $(\mathcal{P}^{n+1}_{(X,M_X)/(S,M_S)})_n$ into a projective system. The two projections $X \times_S X \to X$ give morphism $d_0^n, d_1^n : \mathcal{O}_X \to \mathcal{P}^n_{(X,M_X)/(S,M_S)}$, and multiplication induces a morphism $\pi^n : \mathcal{P}^n_{(X,M_X)/(S,M_S)} \to \mathcal{O}_X$. The maps d_i^n, π^n commute with the transition morphisms $\mathcal{P}^{n+1}_{(X,M_X)/(S,M_S)} \to \mathcal{P}^{n+1}_{(X,M_X)/(S,M_S)}$. Clearly we have

$$\pi^n d_0^n = \pi^n d_1^n = \mathrm{id}_{\mathcal{O}_X} \tag{2.3}$$

There is also an automorphism σ^n of $\mathcal{P}^n_{(X,M_X)/(S,M_S)}$ defined by the automorphism of $X \times_S X$ which "exchanges the entries", which satisfies

$$\sigma^n d_0^n = d_1 \tag{2.4}$$
$$\sigma^n d_1^n = d_0^n \tag{2.5}$$
$$\pi^n \sigma^n = \pi^n \tag{2.6}$$

and such that the squares

$$\begin{tikzcd}
\mathcal{P}_X^n \otimes_A \mathcal{P}_X^n \arrow[r, "\mathrm{id} \otimes \sigma^n"] & \mathcal{P}_X^n \\
\mathcal{P}_X^{2n} \arrow[u, "\delta^{n,n}"] \arrow[r, "\pi^{2n}"] & A \arrow[u, "d_0^n"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}_X^n \otimes_A \mathcal{P}_X^n \arrow[r, "\sigma^n \otimes \mathrm{id}"] & \mathcal{P}_X^n \\
\mathcal{P}_X^{2n} \arrow[u, "\delta^{n,n}"] \arrow[r, "\pi^{2n}"] & A \arrow[u, "d_1^n"']
\end{tikzcd}$$

commute.

2.1.14 Proposition (compare to [Mon02, 2.3.2]). *There is a morphism of rings*

$$\delta^{n,m} : \mathcal{P}^{n+m}_{(X,M_X)/(S,M_S)} \to \mathcal{P}^{n}_{(X,M_X)/(S,M_S)} \otimes \mathcal{P}^{m}_{(X,M_X)/(S,M_S)}$$

compatible with the transition morphisms (2.2)*, such that the following relations hold:*

$$\delta^{n,m} d_0^{m+n} = q_0^{m,n} d_0^{m+n} \tag{2.7}$$
$$\delta^{m,n} d_1^{m+n} = q_1^{m,n} d_1^{m+n} \tag{2.8}$$
$$(\delta^{m,n} \otimes \mathrm{id}_{\mathcal{P}^q})\delta^{m+n,p} = (\mathrm{id}_{\mathcal{P}^m} \otimes \delta^{n,p})\delta^{m,n+p} \tag{2.9}$$

where $q_0^{m,n}$ is the composition

$$\mathcal{P}^{n+m}_{(X,M_X)/(S,M_S)} \twoheadrightarrow \mathcal{P}^{m}_{(X,M_X)/(S,M_S)} \to \mathcal{P}^{m}_{(X,M_X)/(S,M_S)} \otimes \mathcal{P}^{n}_{(X,M_X)/(S,M_S)}$$

$x \mapsto \bar{x} \mapsto x \otimes 1$. *Similarly, $q_1^{m,n}$ is the composition*

$$\mathcal{P}^{n+m}_{(X,M_X)/(S,M_S)} \twoheadrightarrow \mathcal{P}^{n}_{(X,M_X)/(S,M_S)} \to \mathcal{P}^{m}_{(X,M_X)/(S,M_S)} \otimes \mathcal{P}^{n}_{(X,M_X)/(S,M_S)}$$

$x \mapsto \bar{x} \mapsto 1 \otimes x$.

Moreover, the following relations hold:

$$(\pi^m \otimes \mathrm{id}_{\mathcal{P}^n})\delta^{m,n} : \mathcal{P}^{n+m}_{(X,M_X)/(S,M_S)} \to \mathcal{P}^{n}_{(X,M_X)/(S,M_S)} \tag{2.10}$$

and

$$(\mathrm{id}_{\mathcal{P}^m} \otimes \pi^m)\delta^{m,n} : \mathcal{P}^{n+m}_{(X,M_X)/(S,M_S)} \to \mathcal{P}^{m}_{(X,M_X)/(S,M_S)} \tag{2.11}$$

are the transition morphisms. □

Proof. We will not boil down the proof given in [Mon02, 2.3.2] to our situation, but only give the idea: The cartesian diagram

$$\begin{tikzcd}
(\Delta^n_X, M^n_X) \times_{(X,M_X)} (\Delta^m_X, M^n_X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (\Delta^m_X, M^m_X) \arrow[d, "p_0^m"] \\
(\Delta^n_X, M^n_X) \arrow[r, "p_1^n"'] & (X, M_X)
\end{tikzcd}$$

defines an exact closed immersion $(X, M_X) \hookrightarrow (\Delta^n_X, M^n_X) \times_{(X,M_X)} (\Delta^m_X, M^m_X)$ given by an ideal I such that $I^{n+m+1} = 0$. Then, by Proposition 2.1.9, there exists a unique morphism

$$(\Delta^n_X, M^n_X) \times_{(X,M_X)} (\Delta^m_X, M^m_X) \to (\Delta^{m+n}_X, M^{n+m}_X),$$

which induces $\delta^{n,m}$ on the level of sheaves. We refer to Proposition 2.2.7 for the full construction of the morphism $\delta^{n,m}$ in the only case which is of interest to us. ■

The upshot of this rather lengthy discussion is the following proposition:

2.1.15 Proposition. *If $(X, M_X) \to (S, M_S)$ is a finite type morphism of fine* log*-schemes, then the sheaf of rings $\mathcal{O}_X$, together with the projective system $(\mathcal{P}^n_{(X,M_X)/(S,M_S)})_n$ with surjective transition morphisms, and together with the maps π^n, d^n_0, d^n_1, $\delta^{n,m}$, σ^n forms a* formal groupoid in the zariski topos on X. *For the definition of a formal category, see Definition D.1.1.* □

Informally, this means that the data we constructed is “good enough” to do differential geometry with. In particular:

2.1.16 Definition. If $(X, M_X) \to (S, M_S)$ is a finite type morphism of fine log-schemes, then define

$$\mathscr{D}_{(X,M_X)/(S,M_S)} = \varinjlim_n \mathcal{H}om(\mathcal{P}^n_{(X,M_X)/(S,M_S)}, \mathcal{O}_X).$$

This is a quasi-coherent sheaf on X, and the map $\delta^{n,m}$ together with the compatibilities proven above, makes $\mathscr{D}_{(X,M_X)/(S,M_S)}$ into a sheaf of (noncommutative) $\mathcal{O}_X$-bi-algebras. □

We could now formally follow [Ber74, Ch. 2] to define n-connections, stratifications etc. in this context, and if the sheaves $\mathcal{P}^n_{(X,M_X)/(S,M_S)}$ are locally free of finite rank, then we formally obtain the habitual result that giving a stratification on an object E is equivalent to giving a $\mathscr{D}_{(X,M_X)/(S,M_S)}$-module structure on it, see Appendix D.

To be more geometric, this will all be done in the sections following Section 2.2 for a smooth k-scheme with a fixed strict normal crossings divisor.

Logarithmic principal parts for a log-smooth morphism

We compute the local structure of the sheaves $\mathcal{P}^n_{(X,M_X)/(S,M_S)}$ in case (X, M_X) is log-smooth over (S, N).

2.1.17 Definition (Compare to [Mon02, 2.2.2]). We keep the notations of Definition 2.1.12. The sheaf $\mathcal{P}^n_{(X,M_X)/(S,M_S)}$ is equipped with a canonical map $\mu_n : M_X \to \mathcal{P}^n_{(X,M_X)/(S,M_S)}$: Locally we have a factorization

$$(X, M_X) \overset{j}{\hookrightarrow} (Z, M_Z) \longrightarrow (U^{\mathrm{int}}, M_U^{\mathrm{int}})$$

as before. Let $\mathcal{J}_n$ denote the ideal associated with the exact closed immersion $j_n : (X, M_X) \hookrightarrow (\Delta^n_X, M^n_X)$. We have a short exact sequence of monoids

$$0 \longrightarrow j_n^{-1}(1 + \mathcal{J}_n) \xrightarrow{\lambda_n} j_n^{-1} M^n_X \xrightarrow{j_n^*} M_X \longrightarrow 0 \tag{2.12}$$

Here λ_n is defined as follows: M^n_X comes with a morphism $\alpha_n : M^n_X \to \mathcal{O}_{\Delta^n_X}$, inducing an isomorphism $\alpha_n^{-1}(\mathcal{O}^\times_{\Delta_X}) \to \mathcal{O}^\times_{\Delta^n_X}$, and thus a map $1 + \mathcal{J}_n \to \alpha_n^{-1}(\mathcal{O}^\times_{\Delta^n_X}) \to M^n_X$. Applying j_n^{-1} defines λ_n.

Let $p_1, p_2 : (\Delta^n_X, M^n_X) \to (X, M_X)$ denote the two projections. For $i = 1, 2$, we get a map $M_X = j_n^{-1} p_i^{-1} M_X \to j_n^{-1} M^n_X$, and the composition

$$M_X = j_n^{-1} p_i^{-1} M_X \longrightarrow j_n^{-1} M^n_X \longrightarrow M_X$$

is the identity for $i = 1, 2$. Thus, using sequence (2.12), we get a morphism of sheaves of monoids

$$\mu_n : M_X \xrightarrow{p_0^* - p_1^*} j_n^{-1}(1 + \mathcal{J}_n) \xrightarrow{\lambda_n} j_n^{-1} M^n_X \to j_n^{-1} \mathcal{O}_{\Delta^n_X} = \mathcal{P}^n_{(X,M_X)/(S,M_S)}$$

It is not hard to show that for varying n, the μ_n are compatible with the exact closed immersions

$$(\Delta^n_X, M^n_X) \hookrightarrow (\Delta^{n+1}_X, M^{n+1}_X).$$

□

We use the maps μ_n to exhibit explicit local generators of $\mathcal{P}^n_{(X,M_X)/(S,M_S)}$ if (X, M_X) is log-smooth over (S, M_S).

2.1.18 Proposition (cf. [Kat89, 6.5]). *Let (S, N) and (X, M_X) be fine* log-*schemes, and $f : (X, M_X) \to (S, N)$ a* log-*smooth morphism. Let $\bar{x}$ be a geometric point of X. Assume that $m_1, \dots, m_r \in M_{X,\bar{x}}$ are elements such that* $\mathrm{dlog}(m_1), \dots, \mathrm{dlog}(m_r)$ *freely generate $\Omega^1_{(X,M_X)/(S,N),\bar{x}}$. Defining $\eta_i := 1 - \mu_n(m_i)$, then $\mathcal{P}^n_{(X,M_X)/(S,N),\bar{x}}$ is freely generated as either left- or right-$\mathcal{O}_{X,\bar{x}}$-module by monomials of degree $\leq n$ in $\eta_1, \dots, \eta_r$.* □

PROOF. We sketch the proof: Consider the factorization

$$(X, M_X) \underset{\text{closed}}{\overset{\text{exact}}{\hookrightarrow}} (Z, M_Z) \xrightarrow{\text{log-étale}} ((X, M_X) \times_{(S,M_S)} (X, M_X))^{\mathrm{int}}$$

(perhaps we have to replace the product by an open subscheme as before, if X is not separated over S). Let $q_i : (Z, M_Z) \to (X, M_X)$ denote the two projections.

As $(X, M_X) \hookrightarrow (Z, M_Z)$ is exact, we have $(q_i^* M_X)_{\bar{x}} \cong M_{Z,\bar{x}}$ (here $\bar{x}$ also denotes the induced geometric point of Z). Thus, after replacing Z by an étale open, and X by its preimage, we may assume that $M_Z = q_i^* M_X$ for $i = 1, 2$. Then $q_i : (Z, M_Z) \to (X, M_X)$ is a strict log-smooth morphism, whence the morphisms of schemes $q_i : Z \to X$ are smooth in the usual sense. If $\alpha : M_Z \to \mathcal{O}_Z$ is the canonical map, then $\alpha(q_1^*(m_i)q_2(m_i)^{-1}) - 1 \in \mathcal{O}_Z$ is a regular system with respect to both $q_i : Z \to X$. The scheme Δ_X^n now is the classical n-th infinitesimal neighborhood of X in Z, and from the classical theory it follows that $\eta_i = f_n^*(\alpha(q_1^*(m_i)q_2(m_i)^{-1} - 1)$ generates $\mathcal{P}^n_{(X,M_X)/(S,M_S)}$ over $\mathcal{O}_X$ via either q_1 or q_2, as claimed. ■

2.1.19 Proposition. *Let (S, N) be a fine* log*-scheme and $f : (X, M_X) \to (Y, M_Y)$ an (S, N)-morphism of fine* log*-schemes. Then there is a canonical morphism*

$$f^{-1}\mathcal{P}^n_{(Y,M_Y)/(S,N)} \to \mathcal{P}^n_{(X,M_X)/(S,N)}$$

of sheaves of rings, compatible with the morphisms $f^{-1}(d_0)^n, f^{-1}(d_1)^n, f^{-1}(\pi^n)$, $f^{-1}(\delta^{m,n})$, $f^{-1}(\sigma^n)$, etc. In other words, f induces a morphism of formal groupoids

$$\alpha_f : (f^{-1}\mathcal{O}_Y, f^{-1}\mathcal{P}^n_{(Y,M_Y)/(S,M_S)}, \ldots) \to (\mathcal{O}_X, \mathcal{P}_{(X,M_X)/(S,M_S)^n}, \ldots),$$

see Example D.1.6.

This induces a morphism of $\mathcal{O}_X$-algebras

$$\beta_f : f^*\mathcal{P}^n_{(Y,M_X)/(S,M_S)} = f^{-1}\mathcal{P}^n_{(Y,M_Y)/(S,M_S)} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \to \mathcal{P}^n_{(X,M_X)/(S,M_S)},$$

(where the tensor product is taken with respect to the right $f^{-1}\mathcal{O}_Y$-structure of $f^{-1}\mathcal{P}^n_{(Y,M_Y)/(S,M_S)}$) which is an isomomorphism if (Y, M_Y) is log*-smooth over (S, N) and f* log*-étale.* □

Proof. The morphism f induces a commutative diagram of fine (S, M_S)-log-schemes

$$\begin{tikzcd} (X, M_X) \arrow[r, hook, "j_{n,X}"] \arrow[d, "f"'] & (\Delta^n_X, M^n_X) \arrow[d, "f_{\Delta^n}"] \\ (Y, M_X) \arrow[r, hook, "j_{n,Y}"] & (\Delta^n_Y, M^n_Y) \end{tikzcd}$$

in which all arrows are uniquely determined. This induces the morphism

$$\alpha_f : f^{-1}\mathcal{P}^n_{(Y,M_Y)/(S,M_S)} = j^{-1}_{n,X} f^{-1}_{\Delta^n} \mathcal{O}_{\Delta^n_Y} \xrightarrow{j^{-1}_{n,X} f^{\#}_{\Delta^n}} j^{-1}_{n,X}\mathcal{O}_{\Delta^n_X} = \mathcal{P}^n_{(X,M_X)/(S,M_S)},$$

and since f^{-1} commutes with tensor products, we get (for example) a commutative diagram

$$\begin{tikzcd} f^{-1}\mathcal{P}^{m+n}_{Y/S} \arrow[r, "\alpha_f"] \arrow[d, "f^{-1}\delta^{m+n}"'] & \mathcal{P}^{m+n}_{X/S} \arrow[d, "\delta^{m+n}"] \\ f^{-1}\mathcal{P}^m_{Y/S} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{P}^n_{Y/S} \arrow[r, "\alpha_f"] & \mathcal{P}^m_{Y/S} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} \end{tikzcd}$$

The same is true for the other morphisms associated with the formal groupoid defined by $(Y, M_Y)/(S, M_S)$.

If (Y, M_Y) is log-smooth over (S, M_S), then étale locally on Y, we are in the situation of Proposition 2.1.18 for Y. To prove that β_f is an isomorphism for f étale, we may assume there are $m_1, \ldots, m_r \in \Gamma(Y, M_Y)$, such that the $\operatorname{dlog} m_i$ freely generate $\Omega^1_{(Y,M_Y)/(S,M_S)}$. If f is log-étale, , then the induced morphism $f^*\Omega^1_{(Y,M_Y)/(S,M_S)} \to \Omega^1_{(X,M_X)/(S,M_S)}$ is an isomorphism, and $\operatorname{dlog} f^*m_i$ generate $\Omega^1_{(X,M_X)/(S,M_S)}$ freely. The claim follows from the description of the sheaves of log-principal parts in Proposition 2.1.18. ∎

2.1.3 Logarithmic structures and compactifications

We start by giving the main example of interest for our purposes:

2.1.20 Definition. Let X be a scheme and $D \subseteq X$ a Cartier divisor on X.

(a) Then D is called *strict normal crossings divisor*, if

- For every $x \in \operatorname{Supp}(D)$, the local ring $\mathcal{O}_{X,x}$ is regular.
- If D_i, $i \in I$, are the irreducible components of D, considered as reduced closed subscheme of X, then $D = \bigcup_{i=1}^n D_i$, i.e. the closed subscheme D is reduced,
- For every $J \subseteq I$, $J \neq \emptyset$, the closed subscheme $\bigcap_{i \in J} D_i$ is regular.

(b) D is called *normal crossings divisor* if étale locally it is a strict normal crossings divisor. □

2.1.21 Example. (a) Let k be a field and $X = \operatorname{Spec} A$ be an affine k-scheme, with sections $x_1, \ldots, x_n \in A$, such that $\Omega^1_{X/k}$ is freely generated by $dx_1, \ldots, dx_n$. Then the closed subset $D := V(x_1 \cdot \ldots \cdot x_r) \subseteq X$, $r \leq n$, is a strict normal crossings divisor. We associate a log-structure on X with this situation: For an étale open $h : U \to X$, we define

$$M_D(U) = \left\{ a \in \mathcal{O}_U \,\middle|\, a|_{h^{-1}(D)} \in \mathcal{O}^\times_{U \setminus h^{-1}(D)} \right\}.$$

In other words: The sections of M_D over U are those functions on U, which are invertible away from D. We will study this example further in Example 2.1.24. Note that a chart $(X, M_D) \to \operatorname{Spec} \mathbb{Z}[\mathbb{N}^r]$ of (X, M_D) is given by the map $\mathbb{N}^r \to M_D(X)$, $at_i \mapsto x_i^a$, if $t_1, \ldots, t_r$ is a basis for $\mathbb{N}^r$. This shows that M_D is coherent and saturated.

(b) More generally, if X is a scheme and $D \subseteq X$ a normal crossings divisor, then étale locally we are in the situation of (a), and since the sheaf of monoids in the definition of a log-scheme is a sheaf on the étale site of X, we get a log-structure associated with (X, D). □

2.1.22 Proposition. *If X is a scheme and D a normal crossings divisor, then the associated* log-*scheme (X, M_D) is fine and saturated.* □

PROOF. We may assume that we are in the situation of Example 2.1.21, (a). Then a global chart for the log-structure is given by $X \to \operatorname{Spec} \mathbb{Z}[\mathbb{N}^r]$, where $\mathbb{N}^r$ is the free monoid on r generators $t_1, \ldots, t_r$, and the map $\mathbb{N}^r \to M_D$ is given by $at_i \mapsto x_i^a$. This is clearly a fine, saturated monoid. ■

While the two log-structures from Example 2.1.21 are the main examples that will be considered in the following sections, it is still worthwile noting that both are instances of a more general class of examples:

2.1.23 Definition.

- If (Y, M_Y) is a log-scheme, and $f : Y \to X$ a morphism of schemes, then we define the *direct image of* M_X to be the log-structure on X given by the fiber product of the diagram of sheaves of monoids

$$\begin{tikzcd} & f_* M_Y \arrow[d, "f_* \alpha_Y"] \\ \mathcal{O}_X \arrow[r] & f_* \mathcal{O}_Y \end{tikzcd}$$

- Let X be a scheme. The *trivial* log-*structure on* X is the pair $(X, \mathcal{O}_X^\times)$.
- Let $X \hookrightarrow \overline{X}$ be an open immersion of schemes and equip $\overline{X}$ with the direct image $M_{\overline{X}}$ of the trivial log-structure of X. In [Ogu11], this is called the *compactifying* log-*structure on* $\overline{X}$. □

We see that the log-structure associated with a normal crossings divisor $D \subseteq X$ as constructed in Example 2.1.21 is the compactifying log-structure associated with the open immersion $X \setminus D \hookrightarrow X$. For a general open immersion, compactifying log-structures can be extremely complicated.

Since Example 2.1.21, (a) will be our main point of interest, we compute the sheaf of principal parts in this case:

2.1.24 Example. Let k be a field equipped with its trivial log-structure, $X = \operatorname{Spec} A$ an affine smooth k-scheme with coordinates $x_1, \ldots, x_n$, and $D := (x_1 \cdot \ldots \cdot x_r)$ a strict normal crossings divisor. Let M_D be the associated fine log-structure on X, and $\Delta : (X, M_D) \to (X, M_D) \times_k (X, M_D)$ be the diagonal map in the category of log-schemes. The underlying scheme of $(X, M_D) \times_k (X, M_D)$ is $X \times_k X$. There is a global chart $(X, M_D) \to \operatorname{Spec} \mathbb{Z}[\mathbb{N}^r]$, given by $t_i \mapsto x_i$, if $t_1, \ldots, t_r$ generates $\mathbb{N}^r$. We compute the factorization from the proof of Proposition 2.1.7: Writing $Q := \mathbb{N}^r \oplus \mathbb{N}^r$, we have a chart

$$(X, M_D) \times_k (X, M_D) = (X \times_k X, p_1^{-1} M_D \oplus p_2^{-1} M_D) \to \operatorname{Spec} \mathbb{Z}[Q],$$

and a chart for the diagonal is given by the map of monoids $h : \mathbb{N}^r \oplus \mathbb{N}^r \to \mathbb{N}^r$, $(t_i, t_j) \mapsto t_i t_j$. Define $Q' = (h^{gp})^{-1}(\mathbb{N}^r)$. Then

$$Q' = \left\{ (t_1^{\alpha_1} \cdot \ldots \cdot t_r^{\alpha_r}, t_1^{\beta_1} \cdot \ldots \cdot t_r^{\beta_r}) | \alpha_i + \beta_i \geq 0, i = 1, \ldots, r \right\}.$$

By Remark 2.1.8 we know that $Z = (A \otimes_k A \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[Q'])$. Writing $U_i := (t_i^{-1}, t_i) \in Q'$, it is not hard to see that

$$A \otimes_k A \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[Q'] \cong \frac{(A \otimes_k A)[U_1^{\pm 1}, \ldots, U_r^{\pm 1}]}{(1 \otimes x_i - U_i(x_i \otimes 1), i = 1, \ldots, r)},$$

and that the factorization $(X, M_D) \to (Z, M_Z)$ is given by multiplication and $U_i \mapsto 1$. □

2.1.4 Étale vs. Zariski topology

In Example 2.1.21 we saw that even in a relatively simple situation it makes a difference whether one considers log-structures with respect to the étale topology or with respect to the Zariski topology. The following two propositions show that in many situations (in almost all situations that will arise in the rest of this work) we will fortunately be able to use the Zariski topology:

2.1.25 Proposition ([GM71, Lemma 1.8.4]). *If X is a scheme and $D \subseteq X$ a* regular *normal crossings divisor, i.e. if the reduced closed subscheme defined by D is regular, then D is a strict normal crossings divisor.* □

2.1.26 Proposition ([dJ96, 2.4]). *If X is a scheme and $D \subseteq X$ a normal crossings divisor, then there exists a blow-up $\phi : X' \to X$ with center on D, such that the inverse image $(\phi^{-1}(D))_{\mathrm{red}}$ is a strict normal crossings divisor.* □

2.2 The log-diagonal and logarithmic principal parts

Throughout this section let X be a smooth, separated scheme of finite type over an algebraically closed field k, with $D = \sum_{i=1}^r D_i \subseteq X$ a strict normal crossings divisor with irreducible components D_i. This defines a fine, saturated logarithmic structure M_D on the scheme X (see Example 2.1.21), and if $\operatorname{Spec} k$ is equipped with its trivial log-structure, then we obtain a log-smooth morphism $(X, M_D) \to \operatorname{Spec} k$.

The goal of this section is to make the construction of the sheaves of principal parts from Section 2.1.2 precise in this special situation, and then to prove Corollary 2.2.9. This will allow us to construct logarithmic differential operators with properties analogous to non-logarithmic differential operators, using the formal theory of [Ber74, Ch. II]; see Appendix D. Of course, in Proposition 2.1.15 we have already sketched that the conclusion of Corollary 2.2.9 holds in the much greater generality of Section 2.1, but for the sake of explicitness, we spell out precisely what this means in the geometric situation of a smooth scheme with a strict normal crossings divisor.

Note that an alternative approach would be to define $\mathscr{D}_{X/k}(\log D)$ as a subsheaf of $\mathscr{D}_{X/k}$ and deriving its properties from the properties of $\mathscr{D}_{X/k}$.

2.2.1 Definition. The definitions of the log-diagonal and log-products are taken from [KS08, Def. 1.1.1].

- Define $(X \times_k X)'_i \to X \times_k X$ as the blow-up of $X \times_k X$ in $D_i \times D_i$, and $(X \times_k X)^\sim_i$ as the complement of the proper transforms of $D_i \times X$ and $X \times D_i$ in $(X \times_k X)'_i$.
- Define $(X \times_k X)^\sim$ as the product over $X \times_k X$ of all the $(X \times_k X)^\sim_i$.
- The diagonal Δ_X induces an immersion

$$\Delta(D) : X \longrightarrow (X \times_k X)^\sim$$

which we call *the* log-*diagonal of* X *with respect to* D.

- As $\Delta(D)$ is an immersion, there is some open subset U of $(X \times_k X)^\sim$, such that $\Delta(D)(X)$ is a closed subscheme of U. Define $I(D)$ to be the sheaf of ideals defining this closed immersion.

- Define the sheaves of *logarithmic principal parts* $\mathcal{P}^n_{X/k}(\log D)$ as

$$\Delta(D)^{-1}\left(\mathcal{O}_U/I(D)^{n+1}\right).$$

- Let $d_0^n, d_i^n : \mathcal{O}_X \to \mathcal{P}^n_{X/k}(\log D)$ denote the maps induced by the projections $X \times_k X \to X$. This way, $\mathcal{P}^n_{X/k}(\log D)$ is both a left- and a right-$\mathcal{O}_X$-module.

- Let π^n denote the canonical surjective morphism $\mathcal{P}^n_{X/k}(\log D) \to \mathcal{O}_X$. □

2.2.2 Remark. In Section 2.1 we defined the log-diagonal in the more general context of log-schemes. □

Now let us see that the objects we have defined are not so unfamiliar after all:

2.2.3 Proposition. *Let* $X = \operatorname{Spec} A$ *be affine with coordinates* $x_1, \dots x_n$*, such that* D_i *is cut out by* x_i*, for* $i = 1, \dots, r$*. Then the following statements are true.*

(a) $(X \times_k X)^\sim$ *is the spectrum of* $\tilde{A}^2 := \frac{(A\otimes_k A)[U_1^{\pm 1},\dots,U_r^{\pm 1}]}{(1\otimes x_i - U_i(x_i\otimes 1), i=1,\dots r)}$.

(b) ([KS08, Lemma 1.1.2]) *The log diagonal is given by*

$$\frac{(A \otimes_k A)[U_1^{\pm 1}, \dots, U_r^{\pm 1}]}{(1 \otimes x_i - U_i(x_i \otimes 1), i = 1, \dots r)} \longrightarrow A, \tag{2.13}$$

with $a \otimes b \mapsto ab$ *and* $U_i \mapsto 1$.

(c) $I(D)/I(D)^2 \cong \Omega^1_{X/k}(\log D)$*, with* $U_i - 1 \mod I(D)^2 = \frac{dx_i}{x_i}$.

(d) $\mathcal{P}^m_{X/k}(\log D)$ *is free for both its left- and right-*$\mathcal{O}_X$*-structure. If* ξ_i *denotes the image of* $1 \otimes x_i - x_i \otimes 1$*, then* $U_i - 1 = \xi_i/x_i$ *and* $\mathcal{P}^n_{X/k}(\log D)$ *is freely generated by the monomials*

$$\left(\frac{\xi_1}{x_1}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{\xi_r}{x_r}\right)^{\alpha_r} \cdot \xi_{r+1}^{\alpha_{r+1}} \cdot \ldots \cdot \xi_n^{\alpha_n}$$

with $\alpha_i \geq 0$ *and* $\sum_i \alpha_i \leq m$.

(e) *The projection maps* $\mathcal{P}^{m'}_{X/k}(\log D) \to \mathcal{P}^m_{X/k}(\log D)$*, for* $m' > m$*, are surjective and map all the monomials of degree* $> m$ *to* 0.

(f) *We have canonical inclusions* $\mathcal{P}^n_{X/k} \hookrightarrow \mathcal{P}^n_{X/k}(\log D)$. □

PROOF. (a) Let J_i be the ideal of $A \otimes_k A$ generated by $1 \otimes x_i$ and $x_i \otimes 1$. $(X \times_k X)'_i$ is then the blowup of $X \times_k X$ in J_i. As $1 \otimes x_i$ and $x_i \otimes 1$ are regular elements, we have

$$(X \times_k X)'_i = \operatorname{Proj}\left(\frac{(A \otimes_k A)[U_i, V_i]}{(V_i(x_i \otimes 1) - U_i(1 \otimes x_i))}\right).$$

This scheme is covered by the two open affine subschemes

$$\operatorname{Spec}\left(\frac{(A \otimes_k A)[\frac{V_i}{U_i}]}{\left(\frac{V_i}{U_i}(x_i \otimes 1) - (1 \otimes x_i)\right)}\right)$$

and

$$\operatorname{Spec}\left(\frac{(A \otimes_k A)[\frac{U_i}{V_i}]}{\left((x_i \otimes 1) - \frac{U_i}{V_i}(1 \otimes x_i)\right)}\right).$$

Removing the proper transforms of $X \times_k D_i$ and $D_i \times_k X$ makes U_i/V_i and V_i/U_i invertible, so the two resulting open affines agree and are $(X \times_k X)^{\sim}_i$.

Now, after taking the fiber product over $X \times_k X$ of all the $(X \times_k X)^{\sim}_i$, we get the formula (2.13).

(b) This is clear, as $\Delta(D)$ is the unique lift of Δ, by the universal property of blow-ups.

(c) $I(D)/I(D)^2$ is generated as a left A-module by elements $1 \otimes f - f \otimes 1$ and $U_i - 1$. Note that $x_i(U_i - 1) = U_i(x_i \otimes 1) - x_i \otimes 1 = 1 \otimes x_i - x_i \otimes 1$ in $\tilde{A}^2$. Writing $dx_i = 1 \otimes x_i - x_i \otimes 1$, the usual proof shows that $I(D)/I(D)^2$ is freely generated by dx_i/x_i, $i = 1, \dots, r$ and dx_i, $i > r$, as claimed.

(d) The same argument as in (c).

(e) Clear.

(f) Clear, because the monomials $\prod_{i=1}^n \xi_i^{\alpha_i}$ of degree $\leq m$ generate the classical sheaf of principal parts $\mathcal{P}^m_{X/k}$. ■

2.2.4 Corollary. *Now let X be not necessarily affine.*

(a) *$\mathcal{P}^m_{X/k}$ agrees with $\mathcal{P}^m_{(X,M_D)/k}$ (2.1.12), if M_D is the fine* log*-structure associated with D (Example 2.1.21), and if k carries the trivial* log*-structure.*

(b) *$\mathcal{P}^m_{X/k} \hookrightarrow \mathcal{P}^m_{X/k}(\log D)$, where locally ξ_i maps to ξ_i.*

(c) *$\Omega^1_{X/k}(\log D) \cong I(D)/I(D)^2$.*

(d) *The diagonal map induces a surjection*

$$\mathcal{P}^m_{X/k}(\log D) \twoheadrightarrow \mathcal{O}_X,$$

which is split by d_0 and d_1. The kernel of this surjection is $I(D)/I(D)^{m+1}$; in particular, if $m = 1$, then the kernel is $\Omega^1_{X/k}(\log D)$. □

Proof. Only (a) needs a proof, but this is Example 2.1.24. ■

2.2.5 Proposition. *Let Y be a smooth k-scheme and $D_Y \subseteq Y$ a strict normal crossings divisor. If $f : X \to Y$ is a morphism such that $f|_{X\setminus D}$ factors through $Y \setminus D_Y$, then f induces a canonical morphism*

$$f^{-1}\mathcal{P}^n_{Y/k}(\log D_Y) \longrightarrow \mathcal{P}^n_{X/k}(\log D),$$

compatible with the canonical projections $\mathcal{P}^n_{Y/k}(\log D_Y) \to \mathcal{P}^m_{Y/k}(\log D_Y)$, for $m \leq n$. We have a commutative diagram

$$\begin{tikzcd} f^{-1}\mathcal{P}^n_{Y/k}(\log D_Y) \arrow[r] & \mathcal{P}^n_{X/k}(\log D) \\ f^{-1}\mathcal{P}^n_{Y/k} \arrow[r] \arrow[u, hook] & \mathcal{P}^n_{X/k} \arrow[u, hook] \end{tikzcd}$$

where the bottom horizontal morphism is the classical one.

If the map on log*-schemes induced by f is* log*-étale, then*

$$f^*\mathcal{P}^n_{Y/k}(\log D_Y) = f^{-1}\mathcal{P}^n_{Y/k} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \to \mathcal{P}^n_{X/k}(\log D_X)$$

is an isomorphism. □

Proof. This follows from the fact that we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Delta(D)"] \arrow[d, "f"'] & \Delta^n_X(D) \arrow[r, hook] \arrow[d] & \Delta_X(D)^{n+1} \arrow[r, hook] \arrow[d] & (X \times_k X)^\sim \\ Y \arrow[r, "\Delta(D)"] & \Delta^n_Y(D) \arrow[r, hook] & \Delta_Y(D)^{n+1} \arrow[r, hook] & (Y \times_k Y)^\sim \end{tikzcd}$$

See Proposition 2.1.19. ■

2.2.6 Remark. An important special case of Proposition 2.2.5 is the following: Let X, Y be smooth k-schemes, $D_X \subseteq X$, $D_Y \subseteq Y$ strict normal crossings divisors, and $f : X \to Y$ a morphism, such that $f|_{X\setminus D_X}$ is a *finite* morphism $X \setminus D_X \to Y \setminus D_Y$. Then the canonical map $f^*\mathcal{P}^n_{X/k}(\log D_X) \to \mathcal{P}^n_{Y/k}(\log D_Y)$ is an isomorphism if f is log-étale; e.g. if $f|_{X\setminus D_X} : X \setminus D_X \to Y \setminus D_Y$ is an étale covering and f tamely ramified with respect to D_Y. □

2.2.7 Proposition. *For each pair $(n, m) \in \mathbb{N}^2$ there is an $\mathcal{O}_X$-linear (with respect to the both $\mathcal{O}_X$-structures) morphism of rings*

$$\delta^{n,m} : \mathcal{P}^{n+m}_{X/k}(\log D) \longrightarrow \mathcal{P}^m_{X/k}(\log D) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D)$$

where the right- and left-$\mathcal{O}_X$ structures of the sheaves of principal parts are used in the tensor product. The map $\delta^{m,n}$ satisfies the following properties:

(a) *$\delta^{m,n} d^{m+n}_i = q^{m,n}_i d^{m+n}_i$, for $i = 0, 1$, where $q^{m,n}_0$ is the composition*

$$\mathcal{P}^{m+n}_{X/k}(\log D) \to \mathcal{P}^m_{X/k}(\log D) \to \mathcal{P}^m_{X/k}(\log D) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D)$$

where the first arrow is the projection, the second $\xi \mapsto \xi \otimes 1$. The map $q^{m,n}_1$ is defined analogously, by using the the map $\xi \mapsto 1 \otimes \xi$.

(b) $(\pi^m \otimes \mathrm{id}_{\mathcal{P}^n_{X/k}(\log D)})\delta^{m,n}$ *and* $(\mathrm{id}_{\mathcal{P}^m_{X/k}(\log D)} \otimes \pi^n)\delta^{m,n}$ *are the canonical projections. (Recall that* $\pi^n : \mathcal{P}^n_{X/k}(\log D) \twoheadrightarrow \mathcal{O}_X$ *denotes the projection).*

(c) *Finally:*

$$(\delta^{m,n} \otimes \mathrm{id}_{\mathcal{P}^r_{X/k}(\log D)})\delta^{m+n,r} = (\mathrm{id}_{\mathcal{P}^m_{X/k}(\log D)} \otimes \delta^{n,r})\delta^{m+n,r}.$$

□

PROOF. This could be proven by using the triple log-product $(X \times_k X \times_k X)^\sim$. This is not hard, once one knows categorial properties of log-products (in the category of log-schemes), see Proposition 2.1.14.

Since we already know that such a map δ exists in the classical setting for $\mathcal{P}^{m+n}_{X/k}$, our proof will be easier. Lets denote the classical map for the sake of this argument by $\gamma^{m,n} : \mathcal{P}^{m+n}_{X/k} \to \mathcal{P}^m_{X/k} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}$. Recall that locally, $\gamma^{n,m}(a \otimes b) = (a \otimes 1) \otimes (1 \otimes b)$.

We claim that there exists a dotted arrow $\delta^{m,n}$ as in the following diagram:

$$\begin{tikzcd}
\mathcal{P}^{m+n}_{X/k} \arrow[r, hook] \arrow[d, "\gamma^{m,n}"'] & \mathcal{P}^{m+n}_{X/k}(\log D) \arrow[d, dotted, "\delta^{m,n}"] \\
\mathcal{P}^m_{X/k} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k} \arrow[r, hook] & \mathcal{P}^m_{X/k}(\log D) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D)
\end{tikzcd}$$

But this is clear: The existence can be read off of the local description given in Proposition 2.2.3; we locally define $\delta^{m,n}$ by sending ξ_i/x_i to $\gamma^{m+n}(\xi_i)/x_i$, and this glues to a global morphism, because $\gamma^{m,n}$ does.

We need to check that they have the desired properties: As the structure maps $d^n_i : \mathcal{O}_X \to \mathcal{P}^n_{X/k}(\log D)$ factor through the analogous structure maps for $\mathcal{P}^n_{X/k}$, property (a) holds. Properties (b) and (c) can be checked locally and then again follow from local descriptions in Proposition 2.2.3. ■

2.2.8 Proposition. *For each* n*, there is an automorphism* σ^n *of* $\mathcal{P}^n_{X/k}(\log D)$ *satisfying the following properties:*

(a) $\sigma^n d^n_0 = d^n_1$ *and* $\sigma^n d^n_1 = d^n_0$.

(b) $\pi^n \sigma^n = \pi^n$

(c) *The diagrams*

$$\begin{tikzcd}
\mathcal{P}^n_{X/k}(\log D) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D) \arrow[r, "\mathrm{id} \otimes \sigma^n"] & \mathcal{P}^n_{X/k}(\log D) \\
\mathcal{P}^{2n}_{X/k}(\log D) \arrow[u, "\delta^{n,n}"] \arrow[r, "\pi^{2n}"] & \mathcal{O}_X \arrow[u, "d^n_0"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{P}^n_{X/k}(\log D) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D) \arrow[r, "\sigma^n \otimes \mathrm{id}"] & \mathcal{P}^n_{X/k}(\log D) \\
\mathcal{P}^{2n}_{X/k}(\log D) \arrow[u, "\delta^{n,n}"] \arrow[r, "\pi^{2n}"] & \mathcal{O}_X \arrow[u, "d^n_1"']
\end{tikzcd}$$

commute. □

Proof. In the non-logarithmic case the maps σ^n are induced by the map $X \times_k X \to X \times_k X$, "$(x, y) \mapsto (y, x)$". This map extends to the log-product $(X \times_k X)^\sim$, and to the infinitesimal neighborhoods of the log-diagonal. In the local description of Proposition 2.2.3, σ^n is given by

$$\begin{aligned} \tilde{A}^2/I(D)^n &\longrightarrow \tilde{A}^2/I(D)^n \\ a \otimes b &\longmapsto b \otimes a \\ U_i &\longmapsto U_i^{-1} \end{aligned}$$

■

2.2.9 Corollary. *With the notations from above, the data*

$$\left(\mathcal{O}_X, \mathcal{P}^n_{X/k}(\log D), d_0^n, d_1^n, \pi^n, \delta^{m,n}, \sigma^n\right)$$

define a formal groupoid *(in the sense of Definition D.1.1) in the zariski topos of X. We thus have at our disposal the whole machinery from [Ber74, Ch. II]. For a summary of the facts relevant to us, see Appendix D.* □

2.3 Logarithmic differential operators

We continue to fix a smooth, separated, finite type k-scheme X and a strict normal crossings divisor $D \subseteq X$.

2.3.1 Definition. The following definitions are special cases of the definitions in Appendix D.3; see in particular Definition D.3.1.

- The *sheaf of logarithmic differential operators of order* $\leq n$ is defined to be the $\mathcal{O}_X$-bimodule

  $$\mathscr{D}^n_{X/k}(\log D) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/k}(\log D), \mathcal{O}_X).$$

  Its sections are called *differential operators on $\mathcal{O}_X$ of order n.*

- The projections $\mathcal{P}^{n+1}_{X/k}(\log D) \to \mathcal{P}^n_{X/k}(\log D)$ induce inclusions

  $$\mathscr{D}^n_{X/k}(\log D) \hookrightarrow \mathscr{D}^{n+1}_{X/k}(\log D).$$

  and we define $\mathscr{D}_{X/k}(\log D) := \varinjlim_n \mathscr{D}^n_{X/k}(\log D)$.

- If E, F are $\mathcal{O}_X$-modules, we write $\mathscr{D}\!\mathit{iff}^n_D(E, F)$ for the $\mathcal{O}_X$-bimodule

  $$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/k}(\log D) \otimes_{\mathcal{O}_X} E, F),$$

  and its sections are called *logarithmic differential operators from E to F.* □

2.3.2 Proposition.

(a) *The maps $\delta^{m,n}$ from Proposition 2.2.7 induce an associative composition*

$$\mathscr{D}^n_{X/k}(\log D) \times \mathscr{D}^m_{X/k}(\log D) \longrightarrow \mathscr{D}^{m+n}_{X/k}(\log D),$$

which makes $\mathscr{D}_{X/k}(\log D)$ into a sheaf of (noncommutative) unitary left- and right-$\mathcal{O}_X$-algebras.

(b) *The inclusion* $\mathcal{P}^n_{X/k} \hookrightarrow \mathcal{P}^n_X(\log D)$ *from Corollary 2.2.4 induces an inclusion* $\mathscr{D}_{X/k}(\log D) \hookrightarrow \mathscr{D}_{X/k}$.

(c) *If* $X = \operatorname{Spec} A$, *and if* $x_1, \dots, x_n$ *is a system of local coordinates such that* D_i *is cut out by* x_i, $i = 1, \dots, r$, *then* $\mathscr{D}_{X/k}(\log D)$ *is generated as a sheaf of left-*$\mathcal{O}_X$*-algebras by*

$$\delta^{(p^m)}_{x_i} := \frac{x_i^{p^m}}{p^m!}\frac{\partial^{p^m}}{\partial x_i^{p^m}} \text{ for } i = 1, \dots, r \text{ and } \partial^{(p^m)}_{x_i} := \frac{1}{p^m!}\frac{\partial^{p^m}}{\partial x_i^{p^m}} \text{ for } i > r,$$

where $p \geq 0$ *is the characteristic of* k, *and* $m \geq 0$. *See Remark 2.3.3 below.*

(d) *We have the composition formulas:*

$$\partial^{(m)}_{x_i}\partial^{(n)}_{x_j} = \begin{cases} \partial^{(n)}_{x_j}\partial^{(m)}_{x_i} & i \neq j \\ \binom{n+m}{n}\partial^{(n+m)}_{x_i} & \text{else} \end{cases} \tag{2.14}$$

$$\partial^{(n)}_{x_i} f = \sum_{\substack{a+b=n\\ a,b\geq 0}} \partial^{(a)}_{x_i}(f)\partial^{(b)}_{x_i} \tag{2.15}$$

for $f \in A$, *and*

$$\delta^{(n)}_{x_i}\delta^{(m)}_{x_j} = \begin{cases} \delta^{(m)}_{x_j}\delta^{(n)}_{x_i} & i \neq j \\ \sum_{n+m \geq k \leq m} \binom{m}{n+m-k}\binom{k}{k-m}\delta^{(k)}_{x_i} & \text{else} \end{cases} \tag{2.16}$$

(e) *We have* $\mathscr{D}^n_{X/k}(\log D) = \mathscr{D}^n_{X/k} \cap \mathscr{D}_{X/k}(\log D)$.

(f) *If* E *is a* $\mathcal{O}_X$*-module, then* $\mathscr{D}\!\mathit{iff}^{\,n}_D(E, E) \subseteq \mathscr{D}\!\mathit{iff}^{\,n}(E, E)$ *is precisely the subset of operators that locally map* $x_i^\ell E$ *to itself for* $i = 1, \dots, r$ *and all* $\ell \geq 0$. □

2.3.3 Remark. The notation

$$\partial^{(p^m)}_{x_i} = \frac{1}{p^m!}\frac{\partial^{p^m}}{\partial x_i^{p^m}}$$

is taken from [Gie75]. Strictly speaking, it does not make sense, but evaluation of $\partial^{p^m}/\partial x_i^{p^m}$ always yields a binomial coefficient as a factor which is divisible by a sufficiently high power of p. □

PROOF. (a) This is a formal consequence of Corollary 2.2.9, using Proposition D.3.4.

(b) This follows from the local description of $\mathcal{P}^n_{X/k}(\log D)$ in Proposition 2.2.3.

(c) This follows from Proposition 2.2.3 (d).

(d) This follows from the fact that $\mathscr{D}_{X/k}(\log D) \subseteq \mathscr{D}_{X/k}$ is a subring and from the fact that the corresponding formulas in $\mathscr{D}_{X/k}$ are well-known.

(e) This follows from (c).

(f) Let $\partial \in \mathscr{D}\!iff^n(E,E)$ be a differential operator. We have a diagram

$$\begin{tikzcd}
E \arrow[rr] \arrow[d, "d_0"'] & & E \\
\mathcal{P}^n_{X/k} \otimes_{\mathcal{O}_X} E \arrow[urr, "\partial"] \arrow[d] & & \\
\mathcal{P}^n_{X/k}(\log D) \otimes_{\mathcal{O}_X} E \arrow[uurr, dotted] & &
\end{tikzcd}$$

Saying that $\partial \circ d_0$ fixes $x_i^\ell E$ for all i, ℓ now precisely means that ∂ extends to the dotted arrow, i.e. that ∂ lies in the image of

$$\mathscr{D}\!iff^n_D(E,E) \longrightarrow \mathscr{D}\!iff^n(E,E).$$

■

2.3.4 Definition. If k is of characteristic $p \geq 0$, then define $\mathscr{D}^{(m)}_{X/k}(\log D)$ to be the subsheaf of $\mathcal{O}_X$-algebras of $\mathscr{D}_{X/k}(\log D)$ generated by operators of order $\leq p^m$, and call this sheaf *the sheaf of operators of level m.* □

2.3.5 Remark.

- Of course this is only an interesting definition if k has characteristic $p > 0$. Otherwise, using the composition formulas (2.14), it follows that $\mathscr{D}^{(m)}_{X/k}(\log D) = \mathscr{D}_{X/k}(\log D)$ for all $m \geq 0$.
- Using "partially divided power structures", P. Berthelot abstractly defines in [Ber96] sheaves of $\mathcal{O}_X$-algebras $\mathscr{D}^{\mathrm{Ber},(m)}_{X/k}$, which he calls sheaf of "differential operators of level m". He proves that there is an isomorphism $\varinjlim_m \mathscr{D}^{\mathrm{Ber},(m)}_{X/k} \xrightarrow{\cong} \mathscr{D}_{X/k}$. Moreover, "our" sheaf $\mathscr{D}^{(m)}_{X/k}$ is the image of $\mathscr{D}^{\mathrm{Ber},(m)}_{X/k}$ under this isomorphism. We will not make use of the abstract sheaves $\mathscr{D}^{\mathrm{Ber},(m)}_{X/k}$, and hence adopt our definition.

  The main difference of $\mathscr{D}^{\mathrm{Ber},(m)}_{X/k}$ and $\mathscr{D}^{(m)}_{X/k}$ is that if $\theta \in \mathscr{D}^{(m)}_{X/k}$, then $\theta^{p^m} = 0$, while this is not necessarily true in $\mathscr{D}^{\mathrm{Ber},(m)}_{X/k}$.
- Similarly, following Berthelot, in [Mon02] abstract $\mathcal{O}_X$-bialgebras

  $$\mathscr{D}^{\mathrm{Ber},(m)}_{X/k}(\log D)$$

  are defined, such that

  $$\varinjlim_m \mathscr{D}^{\mathrm{Ber},(m)}_{X/k}(\log D) \cong \mathscr{D}_{X/k}(\log D),$$

  and "our" $\mathscr{D}^{(m)}_{X/k}(\log D)$ is the image of the abstract sheaf under this isomorphism. □

2.3.6 Proposition. *Let X, Y be smooth, separated, finite type k-schemes, and D_X, D_Y strict normal crossings divisors on X, resp. Y. Let $f : X \to Y$ be a morphism, such that $f^{-1}(D_Y) \subseteq D_X$ and write*

$$f^*\mathscr{D}_{X/k}(\log D_X) = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathscr{D}_{Y/k}(\log D_Y),$$

where the tensor product uses the left-$f^{-1}\mathcal{O}_Y$-structure of $f^{-1}\mathscr{D}_{Y/k}$. Then the following statements are true:

(a) *$f^*\mathscr{D}_{Y/k}(\log D_Y)$ is a $(\mathscr{D}_{X/k}(\log D_X), f^{-1}\mathscr{D}_{Y/k}(\log D_Y))$-bimodule.*

(b) *There is a canonical left-$\mathscr{D}_{X/k}(\log D_X)$- and right-$f^{-1}\mathcal{O}_Y$-linear map*

$$f^\sharp : \mathscr{D}_{X/k}(\log D_X) \to f^*\mathscr{D}_{Y/k}(\log D_Y) \tag{2.17}$$

fitting in the commutative diagram

$$\begin{tikzcd} \mathscr{D}_{X/k} \arrow[rr] & & f^*\mathscr{D}_{Y/k} \\ \mathscr{D}_{X/k}(\log D_X) \arrow[u, hook] \arrow[rr] & & f^*\mathscr{D}_{Y/k}(\log D_Y) \arrow[u, hook] \end{tikzcd} \tag{2.18}$$

where the top horizontal arrow is the classical morphism induced by f.

(c) *If E is a left-$\mathscr{D}_{Y/k}(\log D_Y)$-module, then $f^*E = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}E$ functorially carries a left-$\mathscr{D}_{X/k}(\log D_X)$-structure, and there is a canonical isomorphism of left-$\mathscr{D}_{X/k}(\log D_X)$-modules*

$$f^*E = f^*\mathscr{D}_{Y/k}(\log D_Y) \otimes_{f^{-1}\mathscr{D}_{Y/k}(\log D_Y)} f^{-1}E,$$

where the $(\mathscr{D}_{X/k}(\log D_X), f^{-1}\mathscr{D}_{Y/k}(\log D_Y))$-bimodule structure from (a) is used.

(d) *If f is* log-*étale, then* (2.17) *is an isomorphism.* □

PROOF. For (a) (and also (c)): The left-$\mathscr{D}_{X/k}(\log D_X)$-structure can be easily obtained from Definition D.2.8 using stratifications, but we give an explicit proof below. The right-$f^{-1}\mathscr{D}_{Y/k}(\log D_Y)$ structure on $f^*\mathscr{D}_{X/k}(\log D_X)$ arises by the functoriality of the construction of the tensor product $f^*\mathscr{D}_{Y/k}(\log D_Y)$, which also shows that both structures are compatible.

(b) follows from Proposition 2.2.5: There is a functorial $\mathcal{O}_X$-linear map

$$f^*\mathcal{P}^n_{Y/k}(\log D_Y) \to \mathcal{P}^n_{X/k}(\log \mathscr{D}_X) \tag{2.19}$$

which induces a map of the associated inductive systems. In turn we get an $\mathcal{O}_X$-morphism

$$\underbrace{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/k}(\log D_X), \mathcal{O}_X)}_{=\mathscr{D}^n_{X/k}(\log D_X)} \to \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{P}^n_{Y/k}(\log D_Y), \mathcal{O}_X)$$

Finally, the canonical map

$$\underbrace{f^*\,\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{P}^n_{Y/k}(\log D_Y), \mathcal{O}_Y)}_{=f^*\mathscr{D}^n_{Y/k}(\log D_Y)} \to \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{P}^n_{Y/k}(\log D_Y), \mathcal{O}_Y)$$

is an isomorphism, since the $\mathcal{P}^n_Y(\log D_Y)$ is locally free of finite rank by the smoothness assumption. Passing to the inductive limit over n gives $f^\sharp$. The fact that $f^\sharp$ fits into the diagram (2.18), follows once one checks, that he morphisms from above respect the inclusions $\mathcal{P}^n_{X/k} \subseteq \mathcal{P}^n_{X/k}(\log D_X)$, but this is easy.

This also proves (d), since in the log-étale case, (2.19) is an isomorphism by Proposition 2.2.5.

The morphism $f^\sharp$ allows us to describe the $\mathscr{D}_{X/k}(\log D_X)$-action on f^*E more explicitly: If $a \otimes e$ is a section of f^*E, and $\theta \in \mathscr{D}^0_{X/k}(\log D_Y)$, then $\theta(a \otimes e) = (\theta a) \otimes e$. If $\theta \in \mathscr{D}^n_{X/k}(\log D_X)$, then θ acts via the Leibniz rule:

$$\theta(a \otimes e) = af^\sharp(\theta)(e) + \theta_a(e),$$

where θ_a is the operator given by $\theta_a(b) = \theta(ab) - a\theta(b)$. This defines the $\mathscr{D}_{X/k}(\log D_X)$-action by induction, as the order of θ_a is $\leq n-1$. ∎

2.4 Logarithmic n-connections

We give the main definitions and properties surrounding n-connections in our logarithmic context. Most statements will follow from Corollary 2.2.9 and the general arguments in Appendix D. Again, this can be done in the more general setup of Section 2.1.

We continue to consider the geometric situation of k an algebraically closed base field, and X a smooth, separated, finite type k-scheme with $D \subseteq X$ a strict normal crossings divisor.

2.4.1 Definition. Let E be an $\mathcal{O}_X$-module. A *logarithmic n-connection on E* is a *right*-$\mathcal{O}_X$-linear morphism

$$\nabla : E \longrightarrow E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D),$$

which reduces to the identity modulo the kernel of the projection

$$\pi^n : \mathcal{P}^n_{X/k}(\log D) \to \mathcal{O}_X.$$

If E, E' are $\mathcal{O}_X$-modules with n-connections ∇,∇', then an $\mathcal{O}_X$-linear morphism f is called *horizontal* if the obvious square diagram commutes.

See Definition D.2.3 for a more general notion of n-connections, and Proposition D.2.5 for why this definition is equivalent to the one given here. □

2.4.2 Proposition. *If E is a locally free $\mathcal{O}_X$-module, then there is an exact sequence*

$$0 \to E \otimes_{\mathcal{O}_X} I(D)/I(D)^{n+1} \to E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D) \to E \to 0, \tag{2.20}$$

where we consider $I(D)/I(D)^{n+1}$ and $E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D)$ as $\mathcal{O}_X$-modules via their right-*$\mathcal{O}_X$-structures. Giving a logarithmic n-connection on E is equivalent to giving a right-$\mathcal{O}_X$-linear splitting $E \to E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D)$ of this sequence.* □

PROOF. This is clear, as E is flat, and the sequence

$$0 \longrightarrow I(D)/I(D)^{n+1} \longrightarrow \mathcal{P}^n_{X/k}(\log D) \longrightarrow \mathcal{O}_X \longrightarrow 0$$

exact. ∎

2.4.3 Proposition. *Giving a logarithmic* 1*-connection is equivalent to giving a logarithmic connection* $\nabla' : E \to E \otimes \Omega^1_{X/k}(\log D)$ *in the classical sense, i.e. a* k*-linear map* ∇'*, such that* $\nabla(fe) = f\nabla(e) + e \otimes df$ *for all sections* f *of* $\mathcal{O}_X$*,* e *of* E. □

PROOF. For an abstract point of view in the spirit of Appendix D, see also [Ber74, Lemme 3.2.1].

This is the same proof as [BO78, Prop 2.9]: Given a logarithmic 1-connection ∇, define $\nabla'(e) = \nabla(e) - e \otimes 1 \otimes 1$. Then $\nabla'(e) \in E \otimes_{\mathcal{O}_X} \Omega^1_{X/k}(\log D)$, as ∇ reduces to the identity modulo $I(D)$. A quick calculation using the right-$\mathcal{O}_X$-linearity of ∇ shows

$$\begin{aligned}
\nabla'(fe) &= \nabla(fe) - fe \otimes 1 \otimes 1 \\
&= \nabla(e)(1 \otimes f) - e \otimes f \otimes 1 + e \otimes 1 \otimes f - e \otimes 1 \otimes f \\
&= (1 \otimes f)(\nabla(e) - e \otimes 1 \otimes 1) + e \otimes (1 \otimes f - f \otimes 1) \\
&= (1 \otimes f)\nabla'(e) + e \otimes df \\
&= f\nabla'(e) + e \otimes df,
\end{aligned}$$

because if $\omega \in \Omega^1_{X/k}(\log D)$, then $(1 \otimes f)\omega = (f \otimes 1)\omega$.

Conversely, given $\nabla' : E \to E \otimes_{\mathcal{O}_X} \Omega^1_{X/k}(\log D)$ a logarithmic connection in the classical sense, defining $\nabla(e) := \nabla'(e) + e \otimes 1 \otimes 1$ gives a morphism $E \to E \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/k}(\log D)$, and the above calculation shows that this morphism is right-$\mathcal{O}_X$-linear. By definition it also reduces to the identity modulo $I(D)$. ■

2.4.4 Proposition. *Giving a logarithmic* n*-connection on an* $\mathcal{O}_X$*-module* E *is equivalent to each of the following data:*

(a) *A* $\mathcal{P}^n_{X/k}(\log D)$*-linear morphism*

$$\mathscr{D}^n_{X/k}(\log D) \longrightarrow \mathscr{D}\!\mathit{iff}^{\,n}_D(E, E).$$

(b) *If* E *is torsion free, a left-*$\mathcal{O}_X$*-linear morphism*

$$\mathscr{D}^n_{X/k}(\log D) \longrightarrow \mathscr{E}\!\mathit{nd}_k(E)$$

satisfying for any open $U \subseteq X$*, and for any* $\partial \in \mathscr{D}^n_{X/k}(\log D)(U)$*,* $f \in \mathcal{O}_X(U)$*, the* Leibniz rule

$$\nabla(\partial)(fe) = f\nabla(\partial)(e) + \nabla(\partial_f)(e)$$

where ∂_f *is the differential operator* $[\partial, f]$.

If E *is not torsion free, a logarithmic* n*-connection still gives rise to the data of* (b). □

PROOF. (a) is Proposition D.3.8, since the kernel of the projection

$$\mathcal{P}^n_{X/k}(\log D_X) \to \mathcal{O}_X$$

is nilpotent. Note that the data of (a) gives the data of (b), as any differential operator $E \to E$ satisfies the Leibniz rule, see e.g. [EGA4, 16.8.8].

Conversely, given the data of (b), we see that the image of ∇ lies in the subring $\mathscr{D}\!\mathit{iff}^n_{X/k}(E,E)$: An endomorphism h of E is a differential operator of order $\leq n$ if and only if $[h,f]$ is a differential operator of order $\leq n-1$ for all $f \in \mathcal{O}_X(U)$. To see that the image is contained in $\mathscr{D}\!\mathit{iff}^n_D(E,E)$ note that the Leibniz rule implies that $\nabla(\partial)$ maps $x_i^k E$ to $x_i^k E$ for every k, $i = 1, \dots r$, where $x_1, \dots, x_r$ are defining equations of $D_1, \dots, D_r$. Hence, by Proposition 2.3.2 (f), the image of ∇ is contained in $\mathscr{D}\!\mathit{iff}^n_D(E,E)$.

It remains to show that ∇ is $\mathcal{P}^n_{X/k}(\log D)$-linear. The morphism

$$\nabla : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/k}(\log D), \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/k}(\log D) \otimes E, E)$$

is additive and left-$\mathcal{O}_X$-linear by assumption. Let f be a section of $\mathcal{O}_X$, then $(1 \otimes f) \cdot \partial = \partial \circ m_{1\otimes f} = \partial_f + f\partial$, where $m_{1\otimes f}$ is multiplication by $1 \otimes f$ in $\mathcal{P}^n_{X/k}(\log D)$. Then we have

$$\nabla((1 \otimes f)\partial)(e) = \nabla(\partial_f)(e) + f\nabla(\partial)(e) = \nabla(\partial)(fe).$$

This shows $\nabla((1 \otimes f)\partial) = \nabla(\partial) \circ m_{1\otimes f} = (1 \otimes f)\nabla(\partial)$, and hence ∇ commutes with elements of the subring generated by $g \otimes f$ (that is $\mathcal{P}^n_{X/k}$!). Now if x_i is a defining equation of D_i, then $\xi_i = 1 \otimes x_i - x_i \otimes 1$ is in this subring, and we have $x_i\nabla(\xi/x_i\partial) = \xi_i\nabla(\partial)$, and as E is torsion free, this implies $\xi_i/x_i\nabla(\partial) = \nabla(\xi/x_i\partial)$ which proves the claim. ■

2.4.5 Proposition. *Let X, Y be smooth separated finite type k-schemes and D_X, D_Y strict normal crossings divisors on X and Y. Let $f : X \to Y$ be a morphism such that $f(X \setminus D_X) \subseteq Y \setminus D_Y$. If E is an $\mathcal{O}_X$-module with logarithmic n-connection with respect to D_X, then f^*E is an $\mathcal{O}_Y$-module with logarithmic n-connection with respect to D_Y.* □

Proof. This follows from Proposition 2.2.5: We have morphisms of abelian sheaves

$$f^{-1}E \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \xrightarrow{f^{-1}\nabla\otimes \mathrm{id}} f^{-1}\mathcal{P}^n_{Y/k} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \to \mathcal{P}^n_{X/k}$$

(note that the n-connection ∇ *right*-$\mathcal{O}_Y$-linear, but not left-$\mathcal{O}_Y$-linear) and the composition is a logarithmic n-connection on f^*E.

Of course on can also just appeal to the construction in Definition D.2.8. ■

2.4.6 Proposition. *Let E, E' be $\mathcal{O}_X$-modules with logarithmic n-connections ∇, ∇'. Then $E \otimes_{\mathcal{O}_X} E'$ and $\mathcal{H}om_{\mathcal{O}_X}(E, E')$ carry logarithmic n-connections. If Y is a smooth, separated, finite type k-scheme, $D_Y \subseteq Y$ a strict normal crossings divisor, and $f : X \to Y$ a morphism such that $f(X \setminus D_X) \subseteq Y \setminus D_Y$, then the canonical maps*

$$f^* \mathcal{H}om_{\mathcal{O}_Y}(E, E') \to \mathcal{H}om_{\mathcal{O}_X}(f^*E, f^*E')$$

is a horizontal morphism, and

$$f^*(E \otimes_{\mathcal{O}_Y} E') \cong (f^*E) \otimes_{\mathcal{O}_X} (f^*E')$$

a horizontal isomorphism.

Moreover, category of $\mathcal{O}_X$-modules with logarithmic n-connection and horizontal morphisms is abelian, and formation of kernels, and cokernels commutes with "forgetting the n-connection". □

PROOF. This follows from Corollary 2.2.9 and Proposition D.2.7. ∎

2.4.7 Definition. A logarithmic n-connection $\nabla : E \to E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D)$ is called *very flat or very integrable* if the induced $\mathcal{O}_X$-linear morphism $\theta : \mathscr{D}^n_{X/k}(\log D) \to \mathscr{E}nd_k(E)$ from Proposition 2.4.4 is a morphism of Lie algebras, and if it is compatible with composition in the following sense: If $\partial_1, \dots, \partial_r \in \mathscr{D}^n_{X/k}(\log D)$ are operators, such that $\prod_i \partial_i \in \mathscr{D}^n_{X/k}(\log D)$, then $\theta(\prod_i \partial_i) = \prod_i \theta(\partial)_i$. □

2.4.8 Remark.

- Note that a 1-connection is very flat if and only if it is flat of p-curvature 0. We have made the awkward choice of words "*very flat*" to distinguish our notion of flatness from the usual notion of integrability.
- A very flat n-connection $\mathscr{D}^n_{X/k}(\log D) \to \mathscr{E}nd_k(E)$ on a torsion free $\mathcal{O}_X$-module E extends uniquely to, and is determined by, a morphism $\mathcal{O}_X$-algebras from the subalgebra of $\mathscr{D}_{X/k}(\log D)$ generated by operators of order $\leq n$ to $\mathscr{E}nd_k(E)$. Consequently, if $p^m \leq n < p^{m+1}$, then the datum of a very flat n-connection on a torsion free $\mathcal{O}_X$-module E is equivalent to a morphism of $\mathcal{O}_X$-algebras $\mathscr{D}^{(m)}_{X/k} \to \mathscr{E}nd_k(E)$.
- Expanding on Remark 2.3.5, there is a notion of p-curvature for $\mathscr{D}^{\mathrm{Ber},(m)}_{X/k}$-modules, see [LSQ97, Def. 3.1.1]. □

2.5 Exponents of logarithmic n-connections

In this section let k be an algebraically closed field of characteristic $p > 0$, X a smooth d-dimensional k-variety, $D \subseteq X$ a strict normal crossings divisor, such that $D = \sum_{i=1}^r D_i$, with D_i smooth divisors.

Recall that we defined the $\mathcal{O}_X$-subalgebras $\mathscr{D}^{(m)}_{X/k} \subseteq \mathscr{D}_{X/k}$ as the subalgebras generated by operators of order $\leq p^m$, and similarly for $\mathscr{D}_{X/k}(\log D)$, and that by Remark 2.4.8 the datum of a very flat logarithmic n-connection on E is equivalent to the datum of a morphism $\mathscr{D}^{(m)}_{X/k}(\log D) \to \mathscr{E}nd_k(E)$ of $\mathcal{O}_X$-algebras, if $p^m \leq n < p^{m+1}$.

2.5.1 Proposition (compare to [Gie75, Lemma 3.8]). *Assume that X is a smooth, separated, finite type k-scheme, and D a smooth divisor. Let $i : D \hookrightarrow X$ be the associated closed immersion and for every $m \geq 0$ write*

$$\overline{\mathscr{D}}^{(m)} := \ker(i^*\mathscr{D}^{(m)}_{X/k}(\log D) \to i^*\mathscr{D}^{(m)}_{X/k}),$$

where the left-$\mathcal{O}_X$-structures of $\mathscr{D}^{(m)}_{X/k}$ and $\mathscr{D}^{(m)}_{X/k}(\log D)$ are used.

If E is a torsion free $\mathcal{O}_X$-module of finite rank with a very flat (Definition 2.4.7) logarithmic n-connection $\nabla : \mathscr{D}^{(m)}_{X/k}(\log D) \to \mathscr{E}nd_k(E)$, and m the unique integer such that $p^m \leq n < p^{m+1}$, then $\overline{\mathscr{D}}^{(m)}$ acts $\mathcal{O}_D$-linearly on $E|_D$, and there exists a decomposition

$$E|_D = \bigoplus_{\alpha \in \mathbb{Z}/p^{m+1}} F_\alpha, \tag{2.21}$$

such that if x_1 is a local defining equation for D, then $\delta_{x_1}^{(s)}(e) = \binom{\alpha}{s}e$, for $s \leq n$ and any section e of F_α. This decomposition is independent of the choice of local coordinates. □

PROOF. The proof is very similar to the proof of Proposition 1.1.12. An operator $\theta \in \mathscr{D}^{(m)}_{X/k}(\log D)$ acts as a differential operator on $\mathcal{O}_D$, and we get a map $\gamma : i^*\mathscr{D}^{(m)}_{X/k}(\log D) \to \mathscr{D}_{D/k}$. In fact, the canoncial map $i^*\mathscr{D}^{(m)}_{X/k}(\log D) \to i^*\mathscr{D}^{(m)}_{X/k}$ factors as follows:

$$i^*\mathscr{D}^{(m)}_{X/k}(\log D) \xrightarrow{\gamma} \mathscr{D}^{(m)}_{D/k} \hookrightarrow i^*\mathscr{D}_{X/k},$$

as can be seen from the local structures. Thus $\overline{\mathscr{D}}^{(m)}$ is precisely the subsheaf of $i^*\mathscr{D}^{(m)}_{X/k}(\log D)$, which acts trivially on $\mathcal{O}_D$, so we get a $\mathcal{O}_D$-linear action of $\overline{\mathscr{D}}^{(m)}$ on $E|_D$. If $x_1, \ldots, x_d$ are local coordinates, such that $D = (x_1)$, then the ideal $\overline{\mathscr{D}}^{(m)}$ is spanned by $\delta_{x_1}^{(r)}$, for $1 \leq r \leq p^m$, and thus $\overline{\mathscr{D}}^{(m)}$ is a locally free $\mathcal{O}_D$-module generated by monomials

$$(\delta_{x_1}^{(p^{m_1})})^{a_1} \cdot (\partial_{x_2}^{(p^{m_2})})^{a_2} \cdot \ldots \cdot (\partial_{x_d}^{(p^{m_d})})^{a_d} + x_1\mathscr{D}^{(m)}_{X/k}(\log D),$$

with $0 \leq m_i \leq m$, $a_i \geq 0$ and $a_1 > 0$.

Now denote by C_{m+1} the set of maps $\mathbb{Z}/p^{m+1} \to \mathbb{Z}/p$ as in Lemma 1.1.9. We have seen that C_{m+1} is an $\mathbb{F}_p$-algebra and that the elements $h_a \in C_m$, $h_a(b) = \binom{b}{a}$, $a \in \mathbb{Z}/p^{m+1}\mathbb{Z}$, are a basis of this $\mathbb{F}_p$-vector space. We define a map $\phi_{m+1} : C_{m+1} \to \overline{\mathscr{D}}^{(m)}$ of $\mathbb{F}_p$-vector spaces by sending h_a to the class of $\delta_{x_1}^{(a)}$, where we identify $\mathbb{Z}/p^{m+1}$ with the set $\{0, \ldots, p^{m+1}-1\}$ (that this identification is allowed, was again was proven in Lemma 1.1.9). As in Lemma 1.1.9 (f) we see that ϕ_{m+1} is in fact a morphism of $\mathbb{F}_p$-algebras, which in turn gives a map

$$C_{m+1} \to \mathrm{End}_k(E|_D)$$

of $\mathbb{F}_p$-algebras. For $\alpha \in \mathbb{Z}/p^{m+1}$, let $\chi_\alpha \in C_{m+1}$ denote the characteristic function $\chi_\alpha(b) = 0$ if $b \neq a$, $\chi_\alpha(\alpha) = 1$. Then the elements $\phi_{m+1}(\chi_\alpha)$ are commuting orthogonal idempotents in $\mathrm{End}_k(E|_D)$, so we get a decomposition

$$E|_D = \bigoplus_{\alpha \in \mathbb{Z}/p^{m+1}} F_\alpha.$$

To compute the action of $\delta_{x_1}^{(s)}$ on F_α, note that $h_s = \sum_{\alpha \in \mathbb{Z}/p^{m+1}} \binom{\alpha}{s}\chi_\alpha$ in C_{m+1}, so indeed $\delta_{x_1}^{(s)}(e) = \binom{\alpha}{s}e$ for every section e of F_α.

We have now constructed a decomposition as in the claim, but only for an open subset of X which admits coordinates $x_1, \ldots, x_d$, such that $\Omega^1_{X/k}$ is free on dx_i. Thus it remains to show that this decomposition does not depend on the choice of the parameters x_i. Let u be a unit, then ux_1 also cuts out D_1 and is part of a system of parameters. We claim that

$$\delta_{ux_1}^{(s)} - \delta_{x_1}^{(s)} \in x_1\mathscr{D}^{(m)}_{X/k}(\log D)$$

for all $s \leq p^m$. This would imply that the operators $\delta_{ux_1}^{(s)}$ and $\delta_{x_1}^{(s)}$ have the same image in $\overline{\mathscr{D}}_j^{(m)}$, so they woudl give rise to the same decomposition of $E|_D$. We compute for every $t \in \mathbb{N}$:

$$\begin{aligned}\delta^{(s)}_{ux_1}(x_1^t) &= \delta^{(s)}_{ux_1}(u^t x_1^t \cdot u^{-t}) \\ &= \sum_{\substack{a+b=s\\ a,b\geq 0}} \delta^{(a)}_{ux_1}(u^{-t}) \cdot u^t \cdot \binom{t}{b} x_1^t\end{aligned}$$

so

$$(\delta^{(s)}_{ux_1} - \delta^{(s)}_{x_1})(x_1^t) = x_1^t \sum_{\substack{a+b=s\\ a>0}} \underbrace{\binom{t}{b} u^t \delta^{(a)}_{ux_1}(u^{-t})}_{\in (x_1)} \in (x_1^{t+1})$$

which completes the proof. ∎

2.5.2 Definition. Let X be a smooth, separated, finite type k-scheme and $D = \sum D_i$ a strict normal crossings divisor with smooth components D_i on X.

Let E be a coherent torsion free $\mathcal{O}_X$-module of finite rank with very flat logarithmic n-connection ∇. Then by Proposition 2.5.1, if U is an open of X such that $U \cap D_i \neq 0$ and $U \cap D_j = 0$ for $j \neq i$, then we can write

$$E|_{D_i \cap U} = \bigoplus_{\alpha \in \mathbb{Z}/p^{m+1}\mathbb{Z}} F_{i,\alpha},$$

where m is the unique integer such that $p^m \leq n < p^{m+1}$. If $F_{i,\alpha} \neq 0$, then we say that α is an *exponent of E along D_i of multiplicity* $\operatorname{rank} F_{i,\alpha}$. □

With the same method, we already obtained a formal local decomposition, in closed points on the boundary divisor, see Proposition 1.1.12. In fact, we can construct a decomposition into rank 1 bundles.

2.5.3 Proposition (compare to [Gie75, Thm. 3.3]). *Let X be a smooth, finite type k-scheme, $D = \bigcup_i D_i$ a strict normal crossings divisor with D_i the smooth components. Moreover, let E be a locally free coherent $\mathcal{O}_X$-module with very flat logarithmic n-connection $\nabla : \mathscr{D}^{(m)}_{X/k}(\log D) \to \mathscr{E}nd_k(E)$, where m is the integer with $p^m \leq n < p^{m+1}$. Let $P \in D$ a closed point of X, say $P \in D_1 \cap \ldots \cap D_r$. Fixing local coordinates $x_1, \ldots, x_n$ around P such that $D_1 = (x_1), \ldots, D_r = (x_r)$, and an isomorphism $k[\![x_1, \ldots, x_n]\!] \cong \widehat{\mathcal{O}_{X,P}}$, there is an isomorphism*

$$E \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}_{X,P}} \cong \bigoplus_{j=1}^{\operatorname{rank} E} \mathcal{O}(\underbrace{\alpha_{1,j}, \ldots, \alpha_{r,j}, 0, \ldots, 0}_{n \text{ arguments}}), \tag{2.22}$$

for some $\alpha_{ij} \in \mathbb{Z}/p^{m+1}$. □

Proof. This is Proposition 1.1.12 applied to $E \otimes \widehat{\mathcal{O}_{X,P}}$. ∎

The exponents defined in Definition 2.5.2 and the α_{ij} from Proposition 2.5.3 are related. In fact, we can read off the exponents from the formal local decomposition (2.22):

2.5.4 Proposition. *Let E be a locally free $\mathcal{O}_X$-module of finite rank with a logarithmic n-connection ∇ and $P \in D_i$ a closed point. The numbers $\alpha_{ij} \in \mathbb{Z}/p^{m+1}$, $j = 1, \dots, \operatorname{rank} E$ from Proposition 2.5.3 are precisely the exponents of E along D_i, in the sense of Definition 2.5.2.*

The mulitplicity of an exponent α of E along D_i is the number of α_{ij} which are equal to α. □

PROOF. Let $x_1, \dots, x_n$ be local coordinates around P, such that D_i is cut out by x_i. We can read off both the α_{ij}, and the exponents α of E along D_i by the action of $\delta_{x_1}^{(p^r)}$ on $E \otimes k(P)$, and it is clear that this action is the same, whether we think of $E \otimes k(P)$ as $E|_D \otimes k(P)$ or as $E \otimes \widehat{\mathcal{O}_{X,P}} \otimes k(P)$. ■

2.5.5 Remark. Classically, the exponents of a logarithmic connection are the eigenvalues of residue maps. A similar construction can be made in our setup: One defines a quotient sheaf $\mathcal{R}^n(D_i)$ of $\mathcal{P}^n_{X/k}(\log D)$, which is isomorphic to $\nu_{i,*}\mathcal{O}^n_{D_i}$, where $\nu_i : D_i \hookrightarrow X$ is the closed immersion associated with D_i. From this, one obtains residue maps $\operatorname{res}_i^n : \mathcal{P}^n_{X/k}(\log D) \to \nu_{i,*}\mathcal{O}^n_{D_i}$.

If (E, ∇) is a logarithmic n-connection, then composition gives a map

$$\operatorname{res}_i^\nabla : E|_{D_i} \xrightarrow{\nabla} (E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D)) \xrightarrow{\operatorname{id} \otimes \operatorname{res}_i^n} (E|_{D_i})^n,$$

and if $x \in D_i$ is a closed point of D_i, then $\operatorname{res}_i^\nabla$ induces a linear map $\operatorname{res}_i^\nabla(x) : E \otimes k(x) \to (E \otimes k(x))^n$, and one checks that for $0 \le j \le n$, $\delta_{x_i}^{(j)}$ acts on $E|_{D_i}$ as the j-th component of $\operatorname{res}_i^\nabla(x)$. It follows that the eigenvalues of $\delta_{x_i}^{(j)}$ acting on $E \otimes k(x)$ are precisely the eigenvalues of the j-th component of $\operatorname{res}_i^\nabla(x)$. Knowing the eigenvalues of $\delta_{x_i}^{(j)}$ for $0 \le j \le n$ amounts to knowing the exponents of E along x_i by construction of the exponents. □

2.6 Logarithmic stratifications

In this section we come to the main definition of this chapter.

2.6.1 Definition. A left-$\mathscr{D}_{X/k}(\log D)$-structure on an $\mathcal{O}_X$-module E is called *logarithmic stratification* if it extends the $\mathcal{O}_X$-structure of E.

If E, E' are $\mathcal{O}_X$-modules carrying logarithmic stratifications, then an $\mathcal{O}_X$-linear morphism is called *horizontal*, if it is also a morphism of left-$\mathscr{D}_{X/k}(\log D)$-modules. □

2.6.2 Remark. *(a)* Of course, in general, the notion of a stratification is different from the notion of of a $\mathscr{D}_{X/k}$-module, see Section D.2. However, in "smooth situations" these notions agree, see Proposition D.3.8 and Remark D.3.9. In our setup with X a smooth k-variety and D a strict normal crossings divisor, the associated log-scheme is log-smooth over $\operatorname{Spec} k$, equipped with its trivial log-structure.

(b) Just like for logarithmic connections in characteristic 0 (see e.g. [EV86, App. C]), an $\mathcal{O}_X$-coherent $\mathscr{D}_{X/k}(\log D)$-module is not necesarrily torsion free: For example, if $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, $D = (xy)$, and $\mathfrak{m} = (x, y)$ the ideal of the origin, then the natural $\mathscr{D}_{X/k}(\log D)$-structure of $k[x, y]$ induces a $\mathscr{D}_{X/k}(\log D)$-structure on the ideals $\mathfrak{m}^2 = (x^2, y^2) \subseteq \mathfrak{m} \subseteq k[x, y]$,

and thus on the quotient $k[x,y]/\mathfrak{m}^2$ (Proposition 2.6.6), which is a torsion module. On the other hand $\mathfrak{m}$ is torsion free, but not locally free. We will give a criterion for a torsion free $\mathcal{O}_X$-coherent $\mathscr{D}_{X/k}(\log D)$-module to be locally free in Theorem 2.7.7. □

2.6.3 Proposition. *If E is a torsion free $\mathcal{O}_X$-module, then the datum of a $\mathscr{D}_{X/k}(\log D)$-module structure on E is equivalent to the datum of a logarithmic n-connection ∇_n on E for every n, such that the following hold:*

- *The diagram*

$$\begin{tikzcd} & E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D) \arrow[dd] \\ E \arrow[ur, "\nabla_n"] \arrow[dr, "\nabla_m"'] & \\ & E \otimes_{\mathcal{O}_X} \mathcal{P}^m_{X/k}(\log D) \end{tikzcd}$$

  commutes for every $n \geq m$, where the vertical arrow is the canonical projection.

- *The diagram*

$$\begin{tikzcd} E \otimes_{\mathcal{O}_X} \mathcal{P}^{m+n}_{X/k}(\log D) \arrow[r, "\mathrm{id}_E \otimes \delta^{m,n}"] & M \otimes_{\mathcal{O}_X} \mathcal{P}^m_{X/k}(\log D) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D) \\ E \arrow[u, "\nabla_{m+n}"] \arrow[r, "\nabla_n"] & E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/k}(\log D) \arrow[u, "\nabla_m \otimes \mathrm{id}"'] \end{tikzcd}$$

  commutes for every pair (n,m).

The logarithmic n-connections ∇_n are automatically very flat. □

PROOF. This is a consequence of Corollary 2.2.9, Proposition D.3.8 and Proposition D.2.7. ■

2.6.4 Proposition. *If E is an $\mathcal{O}_X$-module which is also a $\mathscr{D}_{X/k}(\log D)$-left-module via an $\mathcal{O}_X$-linear ring homomorphism*

$$\nabla : \mathscr{D}_{X/k}(\log D) \to \mathscr{E}nd_k(E),$$

then the following are true:

(a) *For any $U \subseteq X$, and any $\partial \in \mathscr{D}_{X/k}(\log D)(U)$, $f \in \mathcal{O}_X(U), e \in E(U)$ we have the* Leibniz-Rule*:*

$$\nabla(\partial)(fe) = f\nabla(e) + \nabla(\partial_f)(e)$$

where ∂_f is the differential operator $g \mapsto \partial(fg) - f\partial(g)$.

(b) *If $U \subseteq X$ is an open subscheme with local coordinates $x_1, \ldots, x_n$, such that D_i is defined by x_i, for $i = 1, \ldots, r$, then for any $\partial \in \mathscr{D}_{X/k}(\log D)(U)$, $\nabla(\partial)$ maps $x_i^a E|_U$ to $x_i^a E|_U$, for all $a \geq 0$.*

(c) ∇ *maps* $\mathscr{D}^n_{X/k}(\log D)$ *to* $\mathscr{D}\!\mathit{iff}^{\,n}_D(E,E)$. *If* E *is torsion free, then these maps are* $\mathcal{P}^n_{X/k}(\log D)$*-linear.* $\square$

Proof. (a) This follows from Proposition 2.4.4.

(b) This follows from (a).

(c) This follows from Proposition 2.6.3 and Proposition 2.4.4.

2.6.5 Corollary. *If* E *is a torsion free* $\mathcal{O}_X$*-module, then giving a* $\mathscr{D}_{X/k}(\log D)$*-action on* E *is equivalent to giving compatible* $\mathscr{D}_{X/k}(\log D)^{(m)}$*-actions on* E *for every* $m \geq 0$. $\square$

Proof. This is clear, as $\mathscr{D}^{p^m}_{X/k}(\log D)$ generates $\mathscr{D}_{X/k}(\log D)^{(m)}$ for every m. $\blacksquare$

2.6.6 Proposition. *If* E *and* E' *are two* $\mathcal{O}_X$*-modules with logarithmic stratifications* ∇, ∇', *then* $E \otimes_{\mathcal{O}_X} E'$ *and* $\mathcal{H}om_{\mathcal{O}_X}(E, E')$ *carry logarithmic stratifications. If* Y *is a second smooth, separated, finite type* k*-scheme,* $D_Y \subseteq Y$ *a strict normal crossings divisor, and* $f : X \to Y$ *a morphism such that* $f^{-1}(D_Y) \subseteq D$*, then the canonical map*

$$f^* \mathcal{H}om_{\mathcal{O}_Y}(E, E') \to \mathcal{H}om_{\mathcal{O}_X}(f^*E, f^*E')$$

is a horizontal morphism, and

$$f^*(E \otimes_{\mathcal{O}_Y} E') \cong (f^*E) \otimes_{\mathcal{O}_X} (f^*E')$$

a horizontal isomorphism.

Moreover, the category of $\mathcal{O}_X$*-modules with logarithmic stratification and horizontal morphisms is abelian, and formation of kernels and cokernels commutes with "forgetting the stratification".* $\square$

Proof. This follows from Corollary 2.2.9 and Proposition D.2.7.

2.7 Exponents of logarithmic stratifications

In this section we generalize the results from Section 2.5 to logarithmic stratifications. Since a logarithmic stratification is a compatible sequence of n-connections by Proposition 2.6.3, most of the arguments will just be "passing to the limit". In particular, the exponents of a logarithmic stratification will be elements of $\mathbb{Z}_p$.

Again we fix a smooth, separated, finite type k-scheme X, and $D \subseteq X$ a strict normal crossings divisor with smooth components $D_1, \ldots, D_r$.

2.7.1 Proposition (compare to [Gie75, Lemma 3.8]). *Assume that* D *is a smooth divisor. Let* $i : D \hookrightarrow X$ *be the associated closed immersion, and write*

$$\overline{\mathscr{D}} := \ker(i^*\mathscr{D}_{X/k}(\log D) \to i^*\mathscr{D}_{X/k}),$$

where $\mathscr{D}_{X/k}(\log D) \to \mathscr{D}_{X/k}$ *is the canonical inclusion, and* $i^*(-)$ *is computed with respect to the left-*$\mathcal{O}_X$*-structures.*

If E is a torsion free $\mathcal{O}_X$-module of finite rank together with a logarithmic connection

$$\nabla : \mathscr{D}_{X/k}(\log D) \to \mathscr{E}nd_k(E),$$

then $\overline{\mathscr{D}}$ acts $\mathcal{O}_D$-linearly on $E|_D$, and there exists a decomposition

$$E|_D = \bigoplus_{\alpha \in \mathbb{Z}_p} F_\alpha \tag{2.23}$$

such that if, x_1 is a local defining equation for D, then $\delta_{x_1}^{(s)}(e) = \binom{\alpha}{s} e$ for any section e of F_α. □

PROOF. We use the notations from Lemma 1.1.9; in particular: C_m denotes the $\mathbb{F}_p$-agebra of maps $\mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$, and for $a \in \mathbb{Z}/p^m\mathbb{Z}$, $h_a \in C_m$ is the map $b \mapsto \binom{b}{a}$. First assume that there is a global function x_1, such that $D = (x_1)$. Then for every $m \geq 0$, we get a morphism of $\mathbb{F}_p$-algebras $\phi_m : C_m \to \mathrm{End}_k(E|_D)$, as in the proof of Proposition 2.5.1, by sending h_a to $\delta_{x_1}^{(a)}$, for $a \in \mathbb{Z}/p^m$ (here we again identify $\mathbb{Z}/p^m\mathbb{Z}$ with $\{0, \dots, p^m - 1\}$, which is admissible by Lemma 1.1.9). If we again denote for $\alpha_m \in \mathbb{Z}/p^m$ by χ_{α_m} the characteristic functions in C_m associated with α_m, then the χ_{α_m} give a decomposition

$$E|_D = \bigoplus_{\alpha_m \in Z/p^m} F_{\alpha_m}, \tag{2.24}$$

such that each $\delta_{x_1}^{(a)}$ for $a < p^m$ acts via multiplication by $\binom{\alpha_m}{a}$.

Until now everything was already contained in the proof of Proposition 2.5.1. What is new is that we have to let m vary. We also have a map $\phi_{m+1} : C_{m+1} \to \mathrm{End}_k(E|_D)$, and we get a commutative triangle

$$\begin{tikzcd} C_m \arrow[d, "\rho"'] \arrow[dr] & \\ & \overline{\mathscr{D}} \arrow[r, "\nabla"] & \mathrm{End}_k(E|_D) \\ C_{m+1} \arrow[ur] & \end{tikzcd}$$

where $\rho : C_m \to C_{m+1}$ is simply the composition with reduction modulo p^m. Then for $\alpha_m \in \mathbb{Z}/p^m$, we have

$$\rho(\chi_{\alpha_m}) = \sum_{\substack{\beta \in \mathbb{Z}/p^{m+1} \\ \beta \equiv \alpha_m \mod p^m}} \chi_\beta$$

It follows that the decomposition

$$E|_D = \bigoplus_{\alpha_{m+1} \in Z/p^{m+1}} F_{\alpha_{m+1}}$$

refines the decomposition (2.24). Since E is of finite rank, this process stabilizes, and we finally obtain a decomposition

$$E|_D = \bigoplus_{\alpha \in \mathbb{Z}_p} F_\alpha \tag{2.25}$$

such that $\delta_{x_j}^{(a)}$ acts via multiplication by $\binom{\alpha}{a}$ on F_α. Here we again use the properties of the function $\binom{-}{a}$ proven in Lemma 1.1.9.

Finally, we observe that the decomposition (2.25) does not depend on the choice of the parameter x_1: This follows from the fact that the decompositions (2.24) are independent of the choice of x_1, which was proved in Proposition 2.5.1. ■

2.7.2 Remark. Proposition 2.7.1 marks a major difference to the situation in characteristic 0. In fact, it will imply that a $\mathscr{D}_{X/k}(\log D)$-action extends to a $\mathscr{D}_{X/k}$-action, if the exponents along D are 0, see Proposition 4.2.1. In particular, there are no analogs to connections with "nilpotent residues". Essentially, this is true because of the relation $(\delta_{x_1}^{(m)})^p = \delta_{x_1}^{(m)}$. See also Remark 4.2.2, where an explicit example is computed. □

2.7.3 Definition. The elements $\alpha \in \mathbb{Z}_p$ such that $F_\alpha \neq 0$ in the decomposition (2.23), are called *exponents of* $\overline{E}$ along D. If D is not smooth, but $D = \bigcup_{i=1}^r D_i$, D_i a smooth divisor, then *the exponents of* $\overline{E}$ *along* D_i are defined by restricting $\overline{E}$ to an open set $U \subseteq \overline{X}$ intersecting D_i, but not D_j, for $j \neq i$. The set of exponents of $\overline{E}$ along D_i is denoted by $\mathrm{Exp}_{D_i}(\overline{E})$, and the set of exponents of $\overline{E}$ by $\mathrm{Exp}(\overline{E})$. □

2.7.4 Proposition ([Gie75, Thm. 3.3]). *Let X be a smooth, finite type k-scheme, $D = \sum_i D_i$ a strict normal crossings divisor with D_i smooth. Moreover, let E be a locally free coherent $\mathcal{O}_X$-module with logarithmic stratification*

$$\nabla : \mathscr{D}_{X/k}(\log D) \to \mathscr{E}nd_k(E).$$

If $P \in D$ a closed point of X, say $P \in D_1 \cap \ldots \cap D_r$, then after fixing local coordinates $x_1, \ldots, x_n$ around P such that $D_1 = (x_1), \ldots, D_r = (x_r)$, and thus an isomorphism $\widehat{\mathcal{O}_{X,P}} \cong k[\![x_1, \ldots, x_n]\!]$, there is an isomorphism

$$E \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}_{X,P}} \cong \bigoplus_{j=1}^{\operatorname{rank} E} \mathcal{O}(\underbrace{\alpha_{1,j}, \ldots, \alpha_{r,j}, 0, \ldots, 0}_{n \text{ arguments}}), \tag{2.26}$$

for some $\alpha_{ij} \in \mathbb{Z}_p$. □

PROOF. This is Proposition 1.1.12. ■

Also in the case of logarithmic stratifications, the numbers α_{ij} appearing in (2.26) are precisely the exponents of the stratification:

2.7.5 Proposition. *Let E be a locally free $\mathcal{O}_X$-module of finite rank, with a logarithmic stratification ∇, and $P \in D_i$ a closed point, not contained in D_j for $j \neq i$. The numbers $\alpha_{ij} \in \mathbb{Z}_p$, $j = 1, \ldots, \operatorname{rank} E$ from Proposition 2.7.4 are precisely the exponents of E along D_i, in the sense of Definition 2.7.3.*

The mulitplicity of an exponent α of E along D_i is the number of α_{ij} which are equal to α. □

PROOF. Let $x_1, \ldots, x_n$ be local coordinates around P, such that D_i is cut out by x_i. We can read off both the α_{ij}, and the exponents α of E along D_i by the action of $\delta_{x_1}^{(m)}$ on $E \otimes k(P)$, and it is clear that this action is the same, whether we think of $E \otimes k(P)$ as $E|_D \otimes k(P)$ or as $E \otimes \widehat{\mathcal{O}_{X,P}} \otimes k(P)$. ■

2.7.6 Remark. Since the exponents along D_i only depend on the action of the $\delta_{x_i}^{(m)}$, a slight modification of the proof of Proposition 2.7.1 shows that the condition that $P \notin D_j$ for $j \neq i$ is not necessary. □

From this setup we can derive the following important result due to Gieseker. It is the positive characteristic analogue of a theorem of Gérard and Levelt, more precisely, of a consequence of [GL76, Thm. 3.4].

2.7.7 Theorem (compare to [Gie75, Thm. 3.5]). *Let E be a torsion free $\mathcal{O}_X$-coherent $\mathscr{D}_{X/k}(\log D)$-module and assume that the set of exponents of E along D_i maps injectively to $\mathbb{Z}_p/\mathbb{Z}$ for every i, i.e. if α and α' are exponents of E along D_i such that $\alpha - \alpha' \in \mathbb{Z}$, then $\alpha = \alpha'$. Then E is locally free, if and only if E is reflexive.* □

Proof. Clearly, if E is locally free, then it is reflexive.

Conversely, assume that E is reflexive. If $\dim X = 1$, then any torsion free module is locally free, so we may assume $\dim X \geq 2$. Since $E|_{X\setminus D}$ is a $\mathcal{O}_{X\setminus D}$-coherent $\mathscr{D}_{X/k}$-module, E is locally free over $X \setminus D$, see [BO78, 2.17]. Note that since E is torsion free, it is locally free over an open subset $U \subseteq X$ such that $X \setminus U$ has codimension ≥ 2. Let $j : U \hookrightarrow X$ denote the open immersion, then $E = j_*(E|_U)$, because E is reflexive by assumption. Indeed, $j_*(E|_U)$ is the reflexive hull of E. To prove that E is locally free, it suffices to show that $E \otimes \widehat{\mathcal{O}_{X,P}}$ is free for all closed points $P \in D$. Write $R := k[\![x_1, \dots, x_n]\!]$, and fix an isomorphism $R \cong \widehat{\mathcal{O}_{X,P}}$. Since $E = j_*(E|_U)$, we see that $\operatorname{depth}(E \otimes R) \geq 2$, because P is a point of codimension ≥ 2. We can now apply Proposition 1.1.17. ■

Chapter 3

Regular singular stratified bundles

In this chapter we introduce the main objects of study of this dissertation: Regular singular stratified bundles. If X is a smooth k-variety, and E an $\mathcal{O}_X$-coherent $\mathscr{D}_{X/k}$-module, then regular singularity is a condition on what "happens to the $\mathscr{D}_{X/k}$-structure at infinity", i.e. along a compactification X' of X. Since we can only make this precise if the complement $X' \setminus X$ is a strict normal crossings divisor, we proceed in two steps: First, in Section 3.2, we define regular singularity with respect to a fixed "good" partial compactification, and secondly, in Section 3.3, we let the partial compactification vary.

In both sections, we show that the respective categories have the good properties that one would expect. In particular, the category of regular singular stratified bundles (with or without respect to a fixed partial compactification) is tannakian, and a "topological invariant", i.e. pullback along universal homeomorphisms is an equivalence.

In Section 3.1 we prepare this discussion by recalling mostly well-known facts about stratified bundles, and in the final Section 3.5, we give a "tannakian interpretation" of regular singularity. Of course our approach to defining regular singular stratified bundles is not the only sensible one; thus, in Section 3.4, we present other reasonable notions of regular singularity, and compare them to our definition.

3.1 The category of stratified bundles

We first recall the basic definitions and basic properties of stratified bundles, and of the category of stratified bundles. In all of this section X denotes a smooth, connected, separated k-scheme of finite type, and k an algebraically closed field of characteristic $p > 0$.

3.1.1 Definition. A *stratified bundle* on X is a $\mathscr{D}_{X/k}$-module, which is coherent as an $\mathcal{O}_X$-module via the inclusion $\mathcal{O}_X \hookrightarrow \mathscr{D}_{X/k}$. A morphism of stratified bundles (also called "horizontal morphism") is a morphism of $\mathscr{D}_{X/k}$-modules, so in particular a morphism of $\mathcal{O}_X$-modules. We write $\mathrm{Strat}(X)$ for the category of stratified bundles. The trivial line bundle $\mathcal{O}_X$ together with the canonical

$\mathscr{D}_{X/k}$-action will be denoted by $\mathbb{I}_{X/k}$, and we say that an object of $\mathrm{Strat}(X)$ is *trivial* if it is isomorphic to $\mathbb{I}_{X/k}^{\oplus n}$ for some $n \in \mathbb{N}$. □

Recall the following basic properties:

3.1.2 Proposition. *Let X and Y be smooth, finite type k-schemes and $f : Y \to X$ a morphism.*

(a) *A stratified bundle on X is a locally free $\mathcal{O}_X$-module of finite rank.*

(b) *Pull-back of $\mathcal{O}_X$-modules along f induces a functor*

$$f^* : \mathrm{Strat}(X) \to \mathrm{Strat}(Y).$$

(c) *If f is finite and étale, then push-forward of $\mathcal{O}_Y$-modules along f induces a functor*

$$f_* : \mathrm{Strat}(Y) \to \mathrm{Strat}(X),$$

and f_* is right-adjoint to f^*.

(d) *If $E, E' \in \mathrm{Strat}(X)$, then $E \otimes_{\mathcal{O}_X} E'$ and $\mathcal{H}om_{\mathcal{O}_X}(E, E')$ are stratified bundles in a bifunctorial way.* □

PROOF. (a) is e.g. [BO78, 2.17]. Claim (b) follows from Proposition 2.3.6 (for $D = \emptyset$), while claim (d) follows from Proposition 2.6.6. Finally, (c) follows for example from Example D.2.4, which shows that for f finite étale, $f_*\mathcal{O}_Y$ carries a stratification which is compatible with the $\mathcal{O}_X$-algebra structure. Since f is finite, f_*E is coherent for every $\mathcal{O}_Y$-coherent E, so if E carries a $\mathscr{D}_{Y/k}$-action, then f_*E is in fact a stratified bundle on X. Alternatively (which in the end comes down to the same argument), we could also use the isomorphism $f^\sharp : \mathscr{D}_{Y/k} \to f^*\mathscr{D}_{X/k}$: Recall that $f^*\mathscr{D}_{X/k} = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathscr{D}_{X/k}$ (i.e. the left-$f^{-1}\mathcal{O}_X$-structure of $f^{-1}\mathscr{D}_{X/k}$ is used). The isomorphism $f^\sharp$ is then left-$\mathcal{O}_Y$-linear and right $f^{-1}\mathcal{O}_X$-linear. Thus, if E is an $\mathcal{O}_Y$-coherent left-$\mathscr{D}_{Y/k}$-module, then f_*E is a $\mathcal{O}_X$-coherent left-$f_*\mathscr{D}_{Y/k}$-module. But $f^\sharp$ induces a bi-$\mathcal{O}_X$-linear isomorphsm $f_*\mathscr{D}_Y/k \xrightarrow{\cong} f_*f^*\mathscr{D}_{X/k}$, and then by adjunction we get a $\mathscr{D}_{X/k}$-action on f_*E. This argument also shows that for $F \in \mathrm{Strat}(X)$ and E in $\mathrm{Strat}(Y)$, every horizontal morphism $f^*F \to E$ corresponds to a horizontal morphism $F \to f_*E$, which finishes the proof. ■

3.1.3 Remark. As in the classical situation, the definition of a general direct image functor (even for proper morphisms) is more complicated, see [Ber02, §3.4]. We will not make use of this construction. □

We will be particularly interested in the categorical properties of the k-linear category $\mathrm{Strat}(X)$, and certain subcategories. The most important fact is the following.

3.1.4 Proposition ([SR72, §VI.1]). *With the $\otimes$-structure and inner* Hom*-objects from Proposition 3.1.2, the category* $\mathrm{Strat}(X)$ *is a k-linear tannakian category, and any rational point $a \in X(k)$ gives a neutral fiber functor*

$$\omega_a : \mathrm{Strat}(X) \to \mathrm{Vect}_k, E \mapsto a^*E.$$ □

When studying a single stratified bundle E, the $\otimes$-subcategory generated by it is a very convenient tool:

3.1.5 Definition. Define $\langle E\rangle_{\otimes}$ to be the smallest full tannakian subcategory (see Definition C.1.5) of $\mathrm{Strat}(X)$ containing the stratified bundle $E\in\mathrm{Strat}(X)$. In other words, $\langle E\rangle_{\otimes}$ is the full subcategory of $\mathrm{Strat}(X)$ whose objects are isomorphic to subquotients of objects of the form $P(E,E^{\vee})$, with $P(x,y)\in\mathbb{N}[x,y]$.

If $\omega:\langle E\rangle_{\otimes}\to\mathrm{Vectf}_k$ is a fiber functor, then we denote by $\pi_1(\langle E\rangle_{\otimes},\omega)$ the affine k-group scheme associated via Tannaka duality with the tannakian category $\langle E\rangle_{\otimes}\subseteq\mathrm{Strat}(X)$ and ω. This k-group scheme is also called *the monodromy group of E*. It is of finite type over k by Proposition C.2.1. $\square$

It will be important to understand how the category of stratified bundles behaves under restriction to open subsets. This is fairly straightforward:

3.1.6 Proposition. *Let U be an open dense subscheme of X. The following statements are true:*

(a) *If $E\in\mathrm{Strat}(X)$, then the restriction functor $\rho_{U,E}:\langle E\rangle_{\otimes}\to\langle E|_U\rangle_{\otimes}$ is an equivalence.*

(b) *The restriction functor $\rho_U:\mathrm{Strat}(X)\to\mathrm{Strat}(U)$ is fully faithful.*

(c) *If $\omega:\mathrm{Strat}(U)\to\mathrm{Vect}_k$ is a fiber functor, then the induced morphism of k-group schemes $\pi_1(\mathrm{Strat}(U),\omega))\to\pi_1(\mathrm{Strat}(X),\omega|_{\mathrm{Strat}(X)})$ is faithfully flat.*

(d) *If $\mathrm{codim}_X(X\setminus U)\geq 2$, then $\rho_U:\mathrm{Strat}(X)\to\mathrm{Strat}(U)$ is an equivalence.* $\square$

Proof. Without loss of generality we may assume that X is connected. We first prove (d). Assume $\mathrm{codim}_X(X\setminus U)\geq 2$. Denote by $j:U\hookrightarrow X$ the open immersion. Let E be a stratified bundle on U, in particular a locally free, finite rank $\mathcal{O}_U$-module. Then j_*E is $\mathcal{O}_X$-coherent by the assumption on the codimension ([SGA2, Exp. VIII, Prop. 3.2], note that E is Cohen-Macaulay, since it is locally free and since $\mathcal{O}_X$ is Cohen-Macaulay), and it carries a $\mathscr{D}_{X/k}$-action, since $j_*\mathscr{D}_{U/k}=\mathscr{D}_{X/k}$. Thus it is also locally free. If $\overline{E}'$ is any other locally free extension of E to X, then we get a short exact sequence

$$0\to\overline{E}'\hookrightarrow j_*E\to G\to 0,$$

with G supported on $X\setminus U$. Since X is smooth, we have $\mathcal{H}om_{\mathcal{O}_X}(G,\mathcal{O}_X)=0$ and $\mathscr{E}\mathrm{xt}^1_{\mathcal{O}_X}(G,\mathcal{O}_X)=0$ by [SGA2, Exp. III, Prop. 3.3] (note that $\operatorname{depth}\mathcal{O}_{X,x}=\mathrm{codim}_X(x)$ for all $x\in X$, and thus $\operatorname{depth}_{X\setminus U}\mathcal{O}_X\geq 2$). This implies that there is a canonical isomorphism $\overline{E}'\cong((j_*E)^{\vee})^{\vee}=j_*E$. It follows that the functors j_* and j^* are quasi-inverse to each other, which proves that j^* is an equivalence.

Now to prove (a), let $\mathrm{codim}_X(X\setminus U)$ be arbitrary. Let us first show that $\rho_{U,E}:\langle E\rangle_{\otimes}\to\langle E|_U\rangle_{\otimes}$ is essentially surjective. If F is an object of $\langle E\rangle_{\otimes}$, and F' a subobject of $\rho_U(F)=F|_U$, then $j_*F'\subseteq j_*(F|_U)$. The quasi-coherent $\mathcal{O}_X$-modules j_*F' and $j_*(F|_U)$ carry $\mathscr{D}_{X/k}$-actions, such that $j_*F'\subseteq j_*(F|_U)$ and $F\subseteq j_*(F|_U)$ are sub-$\mathscr{D}_{X/k}$-modules. Then $F'_X:=j_*F'\cap F$ is an $\mathcal{O}_X$-coherent

$\mathscr{D}_{X/k}$-submodule of F extending F'. Thus we have seen that the essential image of ρ_U is closed with respect to taking subobjects.

But this also shows that $F|_U/F'$ can be extended to X: We just saw that F' extends to $F'_X \subseteq F$, and since ρ_U is exact, this means that $(F/F'_X)|_U = F|_U/F'$. It follows that the essential image of ρ_U is closed with respect to taking taking subobjects and quotients.

The objects of $\langle E|_U\rangle_\otimes$ are subquotients of stratified bundles bundles of the form $P(E|_U, (E|_U)^\vee)$, with $P(x, y) \in \mathbb{N}[x, y]$. We can lift all of the objects $P(E|_U, (E|_U)^\vee)$ to X, since $(E^\vee)|_U = (E|_U)^\vee$, so ρ_U is essentially surjective.

It remains to check that $\rho_{U,E} : \langle E\rangle_\otimes \to \langle E|_U\rangle_\otimes$ is fully faithful. Let $F_1, F_2 \in \langle E\rangle_\otimes$. Since $\mathrm{Hom}_{\mathrm{Strat}(X)}(F_1, F_2) = \mathrm{Hom}_{\mathrm{Strat}(X)}(\mathcal{O}_X, F_1^\vee \otimes F_2)$, and similarly over U, we may replace F_1 by $\mathcal{O}_X$. Moreover, we may remove closed subsets of codimension ≥ 2 from X. Let W_i be open neighborhoods of the codimension 1 points of X which do not lie in U, such that $X \subseteq W_i$, and such that each W_i contains precisely one codimension 1 point. Then we may replace X by $\bigcup W_i$, and then handle the W_i separately. Thus we may assume that $X \setminus U$ is a smooth divisor D with generic point η. Next, notice that we may shrink X around η. Thus we may assume that $X = \operatorname{Spec} A$ for some ring A, that F_2 corresponds to a free A-module, say with basis $e_1, \dots, e_n$, and that we have global coordinates $x_1, \dots, x_n$, such that $D = (x_1)$. Then $U = \operatorname{Spec} A[x_1^{-1}]$. Finally assume that $\phi : \mathcal{O}_U \to F_2|_U$ is a morphism of stratified bundles given by $\phi(1) = \sum_{i=1}^n \phi_i e_i$, $\phi_i \in A[x_1^{-1}]$. We get

$$0 = \partial_{x_1}(\phi(1)) = \sum_{i=1}^n \partial_{x_1}(\phi_i)e_i + \phi_i \underbrace{\partial_{x_i}(e_i)}_{\in \mathrm{im}(F_2 \to F_2 \otimes A[x_1^{-1}])}$$

and in particular that $\partial_{x_1}(\phi_i) \in \phi_i A \subseteq A[x_1^{-1}]$. By induction, we assume $\partial_{x_1}^{(a)}(\phi_i) \in \phi_i A$ for every $a \leq m$. But then we compute

$$0 = \partial_{x_1}^{(m)}(\phi(1)) = \sum_{i=1}^n \sum_{\substack{a+b=m \\ a,b \geq 0}} \partial_{x_1}^{(a)}(\phi_i)\partial_{x_1}^{(b)}(e_i),$$

to see that $\partial_{x_1}^{(m)}(\phi_i) \in \phi_i A$, so this holds for all $m \geq 1$. In particular we see that the pole order of $\partial_{x_1}^{(m)}(\phi_i)$ along D is at most the pole order of ϕ_i along D, for all $m \geq 0$. To be precise, we consider the image of ϕ_i in $\mathrm{Frac}\,\widehat{(A_\mathfrak{m})}$ for any closed point on D corresponding to some maximal ideal $\mathfrak{m} \supseteq (x_1)$. After choice of our local coordinates, this field is canonically isomorphic to $K := k(\!(x_1, \dots, x_n)\!)$, and here we can measure the pole order along x_1 via the associated discrete valuation v_{x_1}. Here we also see that if $f \in K$ is an element such that $v_{x_1}(f) < 0$, then there is an $m > 0$, such that $v_{x_1}(\partial_{x_1}^{(m)}(f)) < v_{x_1}(f)$. This shows that $\phi_i \in A$ for every i, so ϕ is defined over all of X.

It also extends uniquely: If ψ_1, ψ_2 are morphisms $\mathcal{O}_X \to F_2$, such that $(\psi_1 - \psi_2) \otimes A[x_1^{-1}] = 0$, then $\psi_1 = \psi_2$, as A is an integral domain and F_2 is torsion free. This finishes the proof of (a).

For (b), let $E, E' \in \mathrm{Strat}(X)$ be two stratified bundles. Then $E, E' \in$

$\langle E \oplus E' \rangle_{\otimes}$, so by (a), we see that

$$\begin{aligned}\operatorname{Hom}_{\mathrm{Strat}(X)}(E, E') &= \operatorname{Hom}_{\langle E \oplus E' \rangle_{\oplus}}(E, E') \\ &= \operatorname{Hom}_{\langle E|_U \oplus E|'_U \rangle_{\otimes}}(E|_U, E'|_U) \\ &= \operatorname{Hom}_{\mathrm{Strat}(U)}(E|_U, E'|_U),\end{aligned}$$

so the restriction functor $\mathrm{Strat}(X) \to \mathrm{Strat}(U)$ is fully faithful, and thus (a) is proved.

To prove (c), let $\omega : \mathrm{Strat}(U) \to \mathrm{Vect}_k$ be a fiber functor. By Proposition C.2.3, to show that the induced morphism of k-group schemes is faithfully flat, it remains to show that for every stratified bundle $E \in \mathrm{Strat}(X)$, every substratified bundle of $E|_U$ extends to X. But we have just seen that restriction induces an equivalence $\langle E \rangle_{\otimes} \to \langle E|_U \rangle_{\otimes}$, so we are done. ∎

3.1.7 Proposition. *Let E be a stratified bundle on X, and let ω be any k-valued fiber functor of $\langle E \rangle_{\otimes}$. Then $\pi_1(\langle E \rangle_{\otimes}, \omega)$ is a finite type k-group scheme, and if ω' is a second k-valued fiber functor for $\langle E \rangle_{\otimes}$, then there exists a (noncanonical) isomorphism $\omega \cong \omega'$.* □

PROOF. The first statement is Proposition C.2.1. For the second statement, note that the functor $\underline{\mathrm{Isom}}^{\otimes}_k(\omega, \omega')$ is a $\pi_1(\langle E \rangle_{\otimes}, \omega)$-torsor, representable by an affine, finite type k-scheme, see Theorem C.3.2. Since k is algebraically closed, $\underline{\mathrm{Isom}}^{\otimes}_k(\omega, \omega')$ has a k-point, and thus is trivial. ∎

The next fact about monodromy groups of stratified bundles is absolutely crucial and due to J. P. P. dos Santos. For a local analog, see Proposition 1.3.3.

3.1.8 Proposition ([dS07, Cor. 12]). *If $E \in \mathrm{Strat}(X)$, and ω is a k-linear fiber functor, then the finite type k-group scheme $\pi_1(\langle E \rangle_{\otimes}, \omega)$ is smooth.* □

PROOF. In [dS07, Cor. 12] this is only written down for the case in which $\pi_1(\langle E \rangle_{\otimes}, \omega)$ is finite over k, but the same proof implies our more general assertion: By [dS07, Thm. 11], the affine group scheme $\pi_1(\mathrm{Strat}(X), \omega)$ is perfect, so its ring of global sections is reduced, and thus any quotient G of $\pi_1(\mathrm{Strat}(X), \omega)$ is reduced. If G is also of finite type over k, then the group scheme G is smooth. ∎

To obtain the important results of [dS07], the following equivalence of categories is exploited:

3.1.9 Theorem (Katz, [Gie75, Thm. 1.3]). *The category $\mathrm{Strat}(X)$ of stratified bundles on X is equivalent to the category $\mathrm{Fdiv}(X)$, which is defined as follows:*

- *Objects of $\mathrm{Fdiv}(X)$ are sequences $(E_n, \sigma_n)_{n \geq 0}$, with E_n a vector bundle on X, and σ_n an isomorphism $\sigma_n : F^* E_{n+1} \xrightarrow{\cong} E_n$, where $F : X \to X$ is the absolute frobenius.*
- *A morphism $\phi : (E_n, \sigma_n)_{n \geq 0} \to (E'_n, \sigma'_n)$ in $\mathrm{Fdiv}(X)$ is a collection of $\mathcal{O}_X$-morphisms $\phi_n : E_n \to E'_n$, such that the obvious diagram*

$$\begin{tikzcd} F^* E_{n+1} \arrow[r, "\sigma_n"] \arrow[d, "F^* \phi_{n+1}"'] & E_n \arrow[d, "\phi_n"] \\ F^* E'_{n+1} \arrow[r, "\sigma'_n"] & E\_n \end{tikzcd}$$

commutes. □

We will not reproduce the proof, but recall how the functor $\mathrm{Strat}(X) \to \mathrm{Fdiv}(X)$ is defined: Recall that $\mathscr{D}^{(n)}_{X/k}$ denotes the sheaf of subalgebras of $\mathscr{D}_{X/k}$ generated by operators of degree $\leq p^n$. For every $n \geq 0$ there is a canonical evaluation morphism $\mathscr{D}^{(n)}_{X/k} \to \mathcal{O}_X$, $\partial \mapsto \partial(1)$; let $\mathscr{D}^{+,(n)}_{X/k}$ denote its kernel. If we define E_n as the subsheaf of E on which $\mathscr{D}^{+,(n)}_{X/k}$ acts trivially, then E_n is in a natural way a $\mathcal{O}_{X^{(n+1)}}$-module, so we can think of it as an $\mathcal{O}_X$-module such that the inclusion $E_n \hookrightarrow E$ is p^{n+1}-linear. We get a sequence of p-linear inclusions $E = E_0 \supseteq E_1 \supseteq \ldots$. Cartier's theorem [Kat70, Thm. 5.1] implies that $F^*E_{n+1} \cong E_n$ and that this functor is an equivalence.

An important consequence is:

3.1.10 Proposition. *If $F^n_{X/k} : X \to X^{(n)}$ is the n-th relative frobenius, then pull-back along $F^n_{X/k}$ is an equivalence*

$$(F^n_{X/k})^* : \mathrm{Strat}(X^{(n)}) \to \mathrm{Strat}(X).$$ □

This implies the following analog to the "topological invariance" of the étale fundamental group ([SGA1, IX.4.10])

3.1.11 Theorem ("Topological invariance of Strat(X)"). *If $f : Y \to X$ is a* universal homeomorphism *of smooth, finite type k-schemes, then f induces an equivalence*

$$f^* : \mathrm{Strat}(X) \to \mathrm{Strat}(Y).$$ □

3.1.12 Remark. Recall that by [EGA4, 18.12.11], the finite type morphism f is a universal homeomorphism if and only if it is finite, purely inseparable (i.e. universally injective) and surjective. □

PROOF. Note that f is a finite morphism of degree p^n for some n, and that there is a morphism $g : X \to Y^{(n)}$, such that $gf = F^n_{Y/k}$ is the relative frobenius. This is clear when X and Y are affine, but f is an affine morphism, so everything glues. Now we can use Proposition 3.1.10 to see that the composition $f^*g^* : \mathrm{Strat}(Y^{(n)}) \to \mathrm{Strat}(Y)$ is an equivalence. But this implies that g^* is essentially surjective and full. It is also faithful, since g is faithfully flat (X and $Y^{(n)}$ are regular and g is finite). Thus g^* is an equivalence, and then f^* is an equivalence. ■

3.2 Regular singularities relative to a good partial compactification

In this section we start discussing the notion of a "regular singular stratified bundle". Roughly, on a smooth, separated, finite type k-scheme X, for a stratified bundle E on X, the notion of regular singularity is a condition on how the bundle E "extends to infinity", i.e. how it extends to normal compactifications of X.

Since normal compactifications can be complicated, we first study extensions of E in an easier setup, namely with respect to "good partial compactifications". We generalize some results from the previous section to the category of regular singular stratified bundles.

3.2.1 Definition.

(a) Let $\overline{X}$ be a smooth, separated, finite type k-scheme, and $X \subseteq \overline{X}$ an open subscheme such that the boundary divisor $D := \overline{X} \setminus X$ has strict normal crossings (Definition 2.1.20). We denote such a situation by $(X, \overline{X})$ and call it a *good partial compactification.*

(b) If $(X, \overline{X})$ is a good partial compactification, then a stratified bundle E on X is called $(X, \overline{X})$*-regular* if it extends to an $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}$-module.

(c) If $(X, \overline{X})$ is a good partial compactification, then a stratified bundle E is called $(X, \overline{X})$*-regular singular* if it extends to a torsion free $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module $\overline{E}$ on $\overline{X}$. □

3.2.2 Remark.

- Clearly, a $(X, \overline{X})$-regular bundle is also $(X, \overline{X})$-regular singular.
- The term "$(X, \overline{X})$-regular" is a possible source of confusion: In some texts, regular singular connections are called "regular connections". We only use the distinction of "regular" and "regular singular" relative to a fixed good partial compactification.
- In Appendix A we study good partial compactifications in more detail. The idea is that we can describe the behaviour of a stratified bundle at infinity by describing its extension behavior to all good partial compactifications. □

We connect the notion of $(X, \overline{X})$-regular singular stratified bundles with the notion of regular singularity from Chapter 1.

3.2.3 Proposition. *Let $(X, \overline{X})$ be a good partial compactification with $\overline{X} \setminus X$ irreducible, and assume that $\overline{X} = \operatorname{Spec} A$. Assume that on A there exists a system of parameters $x_1, \ldots, x_n$ such that $X = \operatorname{Spec} A[x_1^{-1}]$, and $\overline{X} \setminus X = V(x_1)$ with x_1. Let E be a stratified bundle on X corresponding to the* free *$A[x_1^{-1}]$-module $E = \bigoplus e_i A[x_1^{-1}]$. Then the following are equivalent:*

(a) *E is $(X, \overline{X})$-regular singular.*

(b) *For every closed point $P \in \overline{X} \setminus X$, the $k(\!(x_1, \ldots, x_n)\!)$-vector space $E \otimes \operatorname{Frac} \widehat{\mathcal{O}_{\overline{X},P}}$ is regular singular in the sense of Definition 1.1.7.*

(c) *There exists a closed point $P \in \overline{X} \setminus X$ such that $E \otimes \operatorname{Frac} \widehat{\mathcal{O}_{\overline{X},P}}$ is regular singular in the sense of Definition 1.1.7.* □

Proof. Clearly (a) implies (b) which implies (c). Assume (c) holds. Let $f \in A[x_1^{-1}]$, and write

$$\delta_{x_1}^{(m)}(fe_i) = \sum_{j=1}^{n} b_{ij}^{(m)}(f) e_j \in E$$

with $b_{ij}^{(m)}(f) \in A[x_1^{-1}]$. The stratified bundle E extends to a finite type A-module stable under the induced action by $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$ if and only if the poles along x_1 of the elements $b_{ij}^{(m)}(f)$ are uniformly bounded for $n \geq 0$, $j = 1, \dots, n$, $f \in A[x_1^{-1}]$. Indeed, if N is a common bound, then the $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$-module generated by $\bigoplus e_i A$ is contained in $\frac{1}{x_1^N} \bigoplus e_i A$ and hence finitely generated.

Since $A[x_1^{-1}] \subseteq \operatorname{Frac} \widehat{\mathcal{O}_{\overline{X},P}}$, we can read off the pole order of on

$$\operatorname{Frac} \widehat{\mathcal{O}_{\overline{X},P}} \cong k((x_1, \dots, x_n)).$$

But by assumption $E \otimes k((x_1, \dots x_n))$ is regular singular, which means that the pole order of the $b_{ij}^{(m)}(f)$ along x_1 is bounded. This shows that (c) implies (a).■

3.2.4 Example. If $(X, \overline{X})$ is a good partial compactification and $L \in \operatorname{Strat}(X)$, such that L is a line bundle, then L is $(X, \overline{X})$-regular singular. This follows directly from Proposition 1.1.10 taking into account Proposition 3.2.3. □

3.2.1 Exponents

Let $(X, \overline{X})$ be a good partial compactification. In Section 2.5 we defined exponents of torsion free $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$-modules. If E is a $(X, \overline{X})$-regular singular stratified bundle, then by definition it extends to such a torsion free $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$-module of finite rank, but this extension is not unique. Nonetheless, in this section we show that the exponents of the different extensions all agree modulo $\mathbb{Z}$, which allows us to define exponents of a $(X, \overline{X})$-regular singular stratified bundle.

3.2.5 Proposition. *Let $(X, \overline{X})$ be a good partial compactification such that $D := \overline{X} \setminus X$ is smooth. Let E be a $(X, \overline{X})$-regular singular stratified bundle, and let $\overline{E}$ and $\overline{E}'$ two torsion free, $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-modules, such that $\overline{E}'|_X = \overline{E}|_X = E$ as $\mathscr{D}_{X/k}$-modules. Write*

$$\overline{E}|_D = \bigoplus_{\alpha \in \mathbb{Z}_p} F_\alpha \text{ and } \overline{E}'|_D = \bigoplus_{\alpha \in \mathbb{Z}_p} F'_\alpha,$$

as in Proposition 2.7.1, and write $\operatorname{Exp}(\overline{E}) := \{\alpha \in \mathbb{Z}_p | F_\alpha \neq 0\}$, $\operatorname{Exp}(\overline{E}') := \{\alpha \in \mathbb{Z}_p | F'_\alpha \neq 0\}$. *Then the images of* $\operatorname{Exp}(\overline{E})$ *and* $\operatorname{Exp}(\overline{E}')$ *in* $\mathbb{Z}_p/\mathbb{Z}$ *are identical.* □

PROOF. It follows from Proposition 2.7.1 that exponents can be computed after removing a closed subset of codimension ≥ 2 from $\overline{X}$, so we may assume that $\overline{E}$ and $\overline{E}'$ are locally free. Note that both $\overline{E}$ and $\overline{E}'$ are $\mathscr{D}_{\overline{X}/k}(\log D)$-submodules of j_*E, if $j : X \hookrightarrow \overline{X}$ denotes the inclusion. Consider the intersection $G := \overline{E} \cap \overline{E'}$ in j_*E. This is a locally free $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module extending E. So to prove the claim, we may assume that $\overline{E}' \subseteq \overline{E}$ is a sub-$\mathscr{D}_{\overline{X}/k}(\log D)$-module. If P is a closed point on D, then it follows from Proposition 2.7.5, that we can compute the exponents on $\widehat{\mathcal{O}_{\overline{X},P}}$. This way, after choosing appropriate coordinates, we may first reduce to inclusions of the form

$$\mathcal{O}(\alpha', 0, \dots) \subseteq \bigoplus_{i=1}^{r} \mathcal{O}(\alpha_i, 0, \dots)$$

on $k[\![x_1, \ldots, x_n]\!]$, where D is defined by $x_1 = 0$, and then by projecting to the summads we may reduce to consider inclusions of the form

$$\mathcal{O}(\alpha', 0, \ldots) \subseteq \mathcal{O}(\alpha, 0, \ldots).$$

Since we are only interested in the action of $\delta_{x_1}^{(m)}$, we may pass to $R := k(\!(x_2, \ldots, x_n)\!)[\![x_1]\!]$. This is a discrete valuation ring, so its submodules are of the form $x_1^{\ell_1} R$ for some $\ell \geq 0$. It is clear that the exponent of (x_1^ℓ) is $\alpha + \ell$, which completes the proof. ■

3.2.6 Definition. Let $(X, \overline{X})$ be a good partial compactification, such that $D := \overline{X} \setminus X = \bigcup_{i=1}^r D_i$, with D_i smooth divisors. If E is an $(X, \overline{X})$-regular singular bundle on E, let $\overline{E}$ be a torsion free $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module extending E. Then write $\mathrm{Exp}_i(E)$ for the image of the set of exponents of $\overline{E}$ along D_i in $\mathbb{Z}_p/\mathbb{Z}$. The set $\mathrm{Exp}_i(E)$ is independent of the choice of $\overline{E}$ by Proposition 3.2.5 and it is called the *set of exponents of E along D_i*. Finally, write $\mathrm{Exp}_{(X,\overline{X})}(E) = \bigcup_i \mathrm{Exp}_i(E)$. □

3.2.7 Remark. We emphasize that, by definition, the exponents of an $(X, \overline{X})$-regular singular bundle lie in $\mathbb{Z}_p/\mathbb{Z}$, while the exponents of a $\mathcal{O}_{\overline{X}}$-locally free $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$-module lie in $\mathbb{Z}_p$. □

3.2.2 An analog of Deligne's canonical extension

In this section, we study in which ways a given $(X, \overline{X})$-regular singular bundle E can extend to a $\mathcal{O}_{\overline{X}}$-locally free $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$-module.

3.2.8 Definition. Let $(X, \overline{X})$ be a good partial compactification, $D := \overline{X} \setminus X$ the boundary divisor, and $\tau : \mathbb{Z}_p/\mathbb{Z} \to \mathbb{Z}$ a section of the projection $\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}_p/\mathbb{Z}$. If E is an $(X, \overline{X})$-regular singular bundle, then a *τ-extension of E* is a locally free, $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module $\overline{E}^\tau$ such that the exponents of $\overline{E}^\tau$ lie in the image of τ.

In particular, no two exponents of $\overline{E}^\tau$ differ by a nonzero integer. □

3.2.9 Theorem. *If $(X, \overline{X})$ is a good partial compactification, $D := \overline{X} \setminus X$, E an $(X, \overline{X})$-regular singular bundle on X, and $\tau : \mathbb{Z}_p/\mathbb{Z} \to \mathbb{Z}$ a section of the projection, then a τ-extension of E exists and is unique up to isomorphisms which restrict to the identity $E \to E$ on X.* □

PROOF. This proof is an extension of the method of [Gie75, Lemma 3.10].

Write $D = \sum D_i$, with D_i smooth divisors, and η_i for the generic point of D_i. Assume that we know that there are open neighborhoods $\overline{U}_i$ of η_i, such that $\eta_j \notin \overline{U}_i$ for $j \neq i$, and such that there exists a τ-extension of $E|_{\overline{U}_i \cap X}$ for the good partial compactification $(\overline{U}_i \cap X, \overline{U}_i)$. If $\overline{U} := \bigcup_i \overline{U}$, then we can glue to get a τ-extension E_τ of E for $(X, X \cup \overline{U})$. But $\operatorname{codim}_{\overline{X}} \overline{X} \setminus (X \cup \overline{U}) \geq 2$, so by Theorem 2.7.7 there exists a τ-extension of E for $(X, \overline{X})$: If $j : X \cup \overline{U} \hookrightarrow \overline{X}$ is the open immersion, then $j_* E_\tau$ is coherent, reflexive and its exponents map injectively to $\mathbb{Z}_p/\mathbb{Z}$.

Similarly, if $\overline{E}_1$ and $\overline{E}_2$ are τ-extensions of E for $(X, \overline{X})$, and $j : \overline{U} \hookrightarrow \overline{X}$ is an open set which contains all codimension 1 points of $\overline{X}$, then $\overline{E}_i = j_*(\overline{E}_i|_U)$, so it suffices to show that there is an isomorphism $\phi : E_1|_{\overline{U}} \cong E_2|_{\overline{U}}$, such that

$\phi|_X = \mathrm{id}_E$. Again we take $\overline{U} := X \cup \bigcup \overline{U}_i$ as above, and to construct ϕ it suffices to construct such a ϕ on every $\overline{U}_i$.

To summarize: we have shown that we can reduce the theorem to the case where $\overline{X}$ is affine, E free, D smooth and cut out by a single global section, lets call it x. Morover, for the proof we may shrink $\overline{X}$ around the generic point of D.

We first prove the existence of a τ-extension. Let $\overline{E}$ be any torsion free $\mathscr{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module extending the stratified bundle E. Shrinking $\overline{X}$ around the generic point $\eta \in D$, we may assume that $\overline{E}$ is free of rank r. Let $\mathrm{Exp}(\overline{E}) = \{\alpha_1, \dots, \alpha_r\}$ be the set of exponents of $\overline{E}$ along D. Let $b := (b_1, \dots, b_r) \in \mathbb{Z}^r$, such that $b_i = b_j$ whenever $\alpha_i = \alpha_j$. It suffices to construct from $\overline{E}$, perhaps after shrinking $\overline{X}$ around η, an $\mathscr{O}_{\overline{X}}$-free $\mathscr{D}_{\overline{X}/k}(\log D)$-module $\overline{E}^{(b)}$ with exponents

$$\mathrm{Exp}(\overline{E}^{(b)}) = \{\alpha_1 + b_1, \alpha_2 + b_2, \dots, \alpha_r + b_r\}.$$

With a suitable choice of b, $E^{(b)}$ is a τ-extension of E.

We proceed in two steps:

(a) If $a \in \mathbb{Z}_+$ then, after shrinking $\overline{X}$ around η, there exists an $\mathscr{O}_{\overline{X}}$-free $\mathscr{D}_{\overline{X}/k}(\log D)$-module $\overline{E}^{(-a)}$ with exponents

$$\mathrm{Exp}(\overline{E}^{(-a)}) = \{\alpha_1 - a, \dots, \alpha_r - a\},$$

such that $\overline{E}^{(-a)}|_X = E$.

Indeed, we can take $\overline{E}^{(-a)} := \overline{E}(aD) = \overline{E} \otimes_{\mathscr{O}_{\overline{X}}} \mathscr{O}_{\overline{X}}(aD)$, because $\mathscr{O}_{\overline{X}}(aD)$ is defined by x^{-a} if x is the defining equation of D.

(b) From $\overline{E}$ we construct a $\mathscr{O}_{\overline{X}}$-free $\mathscr{D}_{\overline{X}/k}(\log D)$-module $\overline{E}_i$ restricting to E on X, such that

$$\mathrm{Exp}(\overline{E}_i) = \{\alpha_j | \alpha_j \neq \alpha_i\} \cup \{\alpha_i + 1\}.$$

Applying the first step for an appropriate $a \in \mathbb{Z}_+$, and then the second step repeatedly for various i, we obtain a τ-extension.

Let us construct $\overline{E}_i$ as in the second step for $i = 1$. Recall that if $\alpha_i = \sum_{j \geq 0} \alpha_{ij} p^j$ is the p-adic expansion of α_i, then $\binom{\alpha}{p^j} = \alpha_{ij} \mod p$. Define $\overline{E}_1$ as the submodule of $\overline{E}$ generated by $x\overline{E}$ and the images of the linear maps

$$\delta_x^{(p^j)} - \alpha_{1j}\,\mathrm{id} \in \mathrm{End}_k(\overline{E})$$

for $j \geq 0$. Then $\overline{E}_1$ is a $\mathscr{D}_{\overline{X}/k}(\log D)$-submodule of $\overline{E}$, because if we choose local coordinates $x = x_1, x_2, \dots, x_n$, then $\delta_x^{(m)}$ commutes with $\partial_{x_i}^{(m')}$ for all m, m' and $i > 1$. After shrinking $\overline{X}$ around η, $\overline{E}_1$ is free, since it is $\mathscr{O}_{\overline{X}}$-coherent and torsion free. Moreover, it restricts to E so its rank is r.

Assume that $\alpha_1 = \ldots = \alpha_\ell$ for some $\ell \in [1, r]$, and that $\alpha_i \neq \alpha_1$ for $i > \ell$. Let $s_1, \dots, s_r$ be a basis of $\overline{E}$ such that

$$\delta_x^{(m)}(s_i) = \binom{\alpha_i}{m} s_i \mod x\overline{E}$$

and define

$$s_i' := \begin{cases} xs_i & i \in [1,\ell] \\ s_i & i > \ell. \end{cases}$$

Then s_i' is a basis of $\overline{E}'$: In fact, $x\overline{E}$ is free with basis $xs_1, xs_2, \ldots$, and

$$\bigoplus_{i>\ell} s_i \mathcal{O}_{\overline{X}} \subseteq \overline{E}_1$$

because

$$\delta_x^{(p^m)}(s_i) - \alpha_{1m}s_i \equiv (\alpha_{im} - \alpha_{1m})s_i \mod x\overline{E}$$

and for $i > \ell$ and large m, $\alpha_{im} \neq \alpha_{1m}$. Since $\operatorname{rank} \overline{E}_1 = r$ it follows that $s_1', \ldots, s_r'$ is a basis of $\overline{E}_1$. Finally, for $1 \leq i \leq \ell$, we compute

$$\delta_x^{(p^k)}(s_i') = xs_i \cdot \binom{\alpha_i + 1}{p^k} \mod x^2\overline{E} = s_i' \cdot \binom{\alpha_i + 1}{p^k} \mod \overline{E}_1$$

and for $i > \ell$ we get

$$\delta_x^{(p^k)}(s_i) = \binom{\alpha_i}{p^k} s_i \mod x\overline{E} = \binom{\alpha_i}{p^k} s_i \mod \overline{E}_1,$$

so finally we see

$$\operatorname{Exp}(\overline{E}_1) = \{\alpha_1 + 1, \alpha_{\ell+1}, \ldots, \alpha_r\}.$$

This finishes the proof of the existence of a τ-extension.

The unicity of τ-extensions follows from the following lemma:

3.2.10 Lemma. *Let $\overline{X} = \operatorname{Spec} A$ be an affine, smooth k-scheme $x_1, \ldots, x_n$, and $D = V(x_1)$ local coordinates. Define $D := V(x_1)$, $X := \overline{X} \setminus D = \operatorname{Spec} A[x_1^{-1}]$, and let $\tau : \mathbb{Z}_p/\mathbb{Z} \to \mathbb{Z}_p$ be a section of the canonical projection. If E is a free $\mathcal{O}_X$-module of rank r with $\mathscr{D}_{X/k}$-action, and if $\overline{E}_1$, $\overline{E}_2$ are free τ-extensions of E, via $\phi : \overline{E}_1|_X \xrightarrow{\sim} \overline{E}_2|_X$, then there is a $\mathscr{D}_{\overline{X}/k}(\log D_X)$-isomorphim $\overline{E}_1 \to \overline{E}_1$ extending ϕ.* □

PROOF. This argument is an adaption of [AB01, Prop 4.7]. Let M be the free A-module corresponding to $\overline{E}_1$ and $\overline{E}_2$. Denote by

$$\nabla_i : \mathscr{D}_{\overline{X}/k}(\log D) \to \operatorname{End}_k(M)$$

the two $\mathscr{D}_{\overline{X}/k}(\log D)$-actions on M, coming from the actions on $\overline{E}_1, \overline{E}_2$. By Proposition 3.2.5 we know that ∇_1 and ∇_2 have the same exponents $\alpha_1, \ldots, \alpha_r$ along D. Let $s_1, \ldots, s_r$ be a basis of M such that

$$\nabla_1 \left(\delta_{x_1}^{(k)}\right)(s_i) \equiv \binom{\alpha_i}{k} s_i \mod x_1M,$$

and let $s_1', \ldots, s_r'$ be a basis of M, such that

$$\nabla_2 \left(\delta_{x_1}^{(k)}\right)(s_i') \equiv \binom{\alpha_i}{k} s_i' \mod x_1M.$$

We need to check that $\phi(s_i) \in M \subseteq M \otimes_A A[x_1^{-1}]$ for all i. Let $k(x_1)$ be the residue field $A_{(x_1)}/x_1A_{(x_1)}$. Fix $i \in \{1, \ldots, r\}$, and write $\phi(s_i) = \sum_{j=1}^r f_{ij}s_j'$

with $f_{ij} \in A[x_1^{-1}]$. Assume that there is an integer $\ell_i > 0$, such that the maximal pole order of the f_{ij} along x_1 is ℓ_i. Then $x_1^{\ell_i}\phi(s_i) \in M \subseteq M \otimes_A A[x_1^{-1}]$, and also

$$x_1^{\ell_i}\nabla_2\left(\delta_{x_1}^{(k)}\right)(\phi(s_i)) \in M \subseteq M \otimes_A A[x_1^{-1}].$$

Tensoring the equality

$$x_1^{\ell_i}\phi\left(\nabla_1\left(\delta_{x_1}^{(k)}\right)(s_i)\right) = x_1^{\ell_i}\nabla_2\left(\delta_{x_1}^{(k)}\right)(\phi(s_i))$$

with $k(x_1)$, we obtain

$$x_1^{\ell_i}\binom{\alpha_i}{k}\sum_{j=1}^{m} f_{ij}s'_j = x_1^{\ell_i}\sum_{j=1}^{m}\nabla_2\left(\delta_{x_1}^{(k)}\right)(f_{ij}s'_j). \tag{3.1}$$

In the completion of the discrete valuation ring $A_{(x_1)}$, we can write $f_{ij} = \sum_{s=-\ell_i}^{d_{ij}} a_{ijs}x_1^s$, with x_1 not dividing in the a_{ijs}, and $a_{ijs} \neq 0$ for $s = -\ell_i$ and some j. But then we can compute

$$x_1^{\ell_i}\nabla_2\left(\delta_{x_1}^{(k)}\right)(f_{ij}s'_j) = \sum_{s=-\ell_i}^{d_{ij}} a_{ijs}x_1^{s+\ell_i}\binom{\alpha_j+s}{k}s'_j$$

Then (3.1) gives in $k(x_1)$ the equation

$$x_1^{\ell_i+s}a_{ijs}\binom{\alpha_i}{k} = a_{ijs}x_1^{\ell_i+s}\binom{\alpha_j+s}{k},$$ ■

both sides of which are 0, except when $s = -\ell_i$. But this implies $\binom{\alpha_i}{k} = \binom{\alpha_j-\ell_i}{k}$ for all k, and hence $\alpha_i = \alpha_j - \ell_i$ by Lucas' Theorem 1.1.9 (a), which is is impossible, as $\alpha_1, \dots, \alpha_r$ lie in the image of τ, and thus map injectively to $\mathbb{Z}_p/\mathbb{Z}$. Thus $\ell_i = 0$ and hence $\phi(s_i) \in M$. ■

3.2.3 The category of $(X, \overline{X})$-regular singular bundles

In this section we define the category of $(X, \overline{X})$-regular singular stratified bundles for a good partial compactification $(X, \overline{X})$, and generalize some of the results of Section 3.1 to this context.

We begin by observing that the notion of $(X, \overline{X})$-regular singularity is "local in codimension 1 points":

3.2.11 Proposition ([Gie75, Thm. 3.5]). *Let $(X, \overline{X})$ be a good partial compactification and E a stratified bundle on X. Then the following are equivalent.*

(a) *E is $(X, \overline{X})$-regular singular,*

(b) *there exists an open subset $\overline{U}$ of $\overline{X}$, with $\operatorname{codim}_{\overline{X}}(\overline{X} \setminus (\overline{U} \cup X) \geq 2$, such that $E|_{\overline{U}\cap X}$ is $(\overline{U} \cap X, \overline{U})$-regular singular.*

(c) *E extends to a* locally free*, finite rank $\mathscr{O}_{\overline{X}}$-module with $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$-action extending the stratification on E.* □

Proof. (c) clearly implies (a) which also obviously implies (b). Assume (b) holds. Let $\overline{E}$ be a torsion free $\mathcal{O}_{\overline{U}}$-coherent $\mathscr{D}_{\overline{U}/k}(\log \overline{U} \setminus X)$-module extending E. Since $\overline{E}$ is torsion free, there is an open subset $\overline{U}' \subseteq \overline{U}$, such that $X \subseteq \overline{U}'$, and $\operatorname{codim}_{\overline{X}}(\overline{X} \setminus \overline{U}') \geq 2$, on which $\overline{E}$ is locally free. Let $j : \overline{U}' \subseteq \overline{X}$ be the open immersion. Then $j_*(\overline{E}|_{\overline{U}'})$ is an $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$-module, which is also reflexive. But then it is also locally free by Theorem 2.7.7, so (b) implies (c). ■

3.2.12 Proposition. *Let $(X, \overline{X})$ be a good partial compactification, $\eta_1, \ldots, \eta_r$ the generic points of $\overline{X} \setminus X$, and E a stratified bundle on X. Then E is $(X, \overline{X})$-regular (resp. $(X, \overline{X})$-regular singular) if and only if there are open neighborhoods $\overline{U}_i$ of η_i, $i = 1, \ldots, r$, such that $E|_{\overline{U}_i \cap X}$ is $(\overline{U}_i \cap X, \overline{U}_i)$-regular (resp. $(\overline{U}_i \cap X, \overline{U}_i)$-regular singular).* □

Proof. Only the 'if' direction is interesting. Given open neighborhoods $\overline{U}_i$ of η_i, $i = 1, \ldots, r$, we may assume, by shrinking the $\overline{U}_i$ if necessary, that $\eta_j \notin \overline{U}_i$ if $i \neq j$. Then $\overline{U}_i \cap \overline{U}_j \subseteq X$ for $i \neq j$. Hence, if $\overline{E}_i$ is a torsion free $\mathcal{O}_{\overline{U}_i}$-coherent extension of $E|_{\overline{U}_i \cap X}$ to $\overline{U}_i$ as a $\mathscr{D}_{\overline{U}_i/k}$-module (resp. $\mathscr{D}_{\overline{U}_i/k}(\log \overline{U}_i \setminus X)$-module), then we can glue to obtain an extension $\overline{E}'$ on $\bigcup_i \overline{U}_i$. But $\overline{X} \setminus (\bigcup_i \overline{U}_i \cup X)$ has codimension ≥ 2 in $\overline{X}$, so we can apply Proposition 3.1.6, (d) (resp. Proposition 3.2.11) to finish the proof. ■

3.2.13 Definition. If $(X, \overline{X})$ is a good partial compactification, then we define $\operatorname{Strat}^{\mathrm{rs}}((X, \overline{X}))$ to be the full subcategory of $\operatorname{Strat}(X)$ consisting of $(X, \overline{X})$-regular singular bundles. □

3.2.14 Proposition. *Let $(X, \overline{X})$ be a good partial compactification, and E, E' $(X, \overline{X})$-regular (resp. $(X, \overline{X})$-regular singular) bundles. Then the following stratified bundles are also $(X, \overline{X})$-regular (resp. $(X, \overline{X})$-regular singular):*

(a) *Every substratified bundle $F \subseteq E$.*

(b) *Every quotient-stratified bundle E/F of E.*

(c) *$E \otimes_{\mathcal{O}_X} E'$.*

(d) *$\mathcal{H}om_{\mathcal{O}_X}(E, E')$.*

It follows that $\operatorname{Strat}^{\mathrm{rs}}((X, \overline{X}))$ *is a tannakian subcategory (Definition C.1.5) of* $\operatorname{Strat}(X)$. *In particular, if $\iota : \operatorname{Strat}^{\mathrm{rs}}((X, \overline{X})) \hookrightarrow \operatorname{Strat}(X)$ denotes the inclusion functor, then for a $(X, \overline{X})$-regular singular bundle E, restriction of ι gives an equivalence $\langle E \rangle_{\otimes} \to \langle \iota(E) \rangle_{\otimes}$.* □

Proof. Write $D := \overline{X} \setminus X$, $j : X \hookrightarrow \overline{X}$. For (a), let $\overline{E}$ be a torsion free $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}$-module (resp. $\mathscr{D}_{\overline{X}/k}(\log D)$-module) extending E. Then j_*F and $\overline{E}$ are both $\mathscr{D}_{\overline{X}/k}$-submodules (resp. $\mathscr{D}_{\overline{X}/k}(\log D)$-submodules) of j_*E. Let $\overline{F}$ be their intersection. This is an $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}$-module (resp. $\mathscr{D}_{\overline{X}/k}(\log D)$-module) extending $\overline{F}$. But then $\overline{E}/\overline{F}$ is an $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module extending E/F, so (a) and (b) follow.

If $\overline{E}, \overline{E'}$ are $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-modules extending E and E', then (c) and (d) follow from the fact that $\mathcal{H}om_{\mathcal{O}_{\overline{X}}}(\overline{E}, \overline{E}')$ and $\overline{E} \otimes_{\mathcal{O}_{\overline{X}}} \overline{E}'$ carry natural $\mathscr{D}_{\overline{X}/k}(\log D)$-actions extending those coming from E and E', see Proposition 2.6.6. ■

3.2.15 Remark. (a) In fact it follows from Theorem 3.2.9 that the category of $(X, \overline{X})$-regular bundles is equivalent to $\mathrm{Strat}(\overline{X})$. More precisely, it is the essential image of the restriction functor $\mathrm{Strat}(\overline{X}) \to \mathrm{Strat}(X)$.

(b) It is not true that $\mathrm{Strat}^{\mathrm{rs}}((X, \overline{X}))$ is closed under extensions: For example, there are extensions of $\mathcal{O}_X$ by $\mathcal{O}_X$ which are not $(X, \overline{X})$-regular singular: Take $(X, \overline{X}) = (\mathbb{A}^1_k, \mathbb{P}^1_k)$, and let T be the coordinate of $\mathbb{A}^1_k$. Let $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ be the Artin-Schreier covering $T \mapsto T^p - T$. Then $f_*\mathcal{O}_{\mathbb{A}^1_k}$ is free of rank p, and the stratified subbundle spanned by $1, T$ is an extenion of $\mathcal{O}_{\mathbb{A}^1_k}$ by $\mathcal{O}_{\mathbb{A}^1_k}$ which is not regular singular. The proof is the same as in Example 1.4.4, (f). □

3.2.16 Proposition. *Let $(X, \overline{X})$ be a good partial compactification and $U \subseteq X$ an open subscheme. Then:*

(a) *There is an open subscheme $\overline{U} \subseteq \overline{X}$ such that $\mathrm{codim}_{\overline{X}}(\overline{X} \setminus (X \cup \overline{U})) \geq 2$ and such that $(U, \overline{U})$ is a good partial compactification.*

For any $(U, \overline{U})$ as in (a), the following statements are true:

(b) *The restriction functor $\rho_U : \mathrm{Strat}^{\mathrm{rs}}((X, \overline{X})) \to \mathrm{Strat}^{\mathrm{rs}}((U, \overline{U}))$ is a fully faithful, exact $\otimes$-functor.*

(c) *If $E \in \mathrm{Strat}^{\mathrm{rs}}((X, \overline{X}))$, then ρ_U induces an equivalence $\langle E \rangle_{\otimes} \xrightarrow{\sim} \langle E|_U \rangle_{\otimes}$.*

(d) *If $\omega : \mathrm{Strat}^{\mathrm{rs}}((U, \overline{U})) \to \mathrm{Vectf}_k$ is a neutral fiber functor, then the morphism of k-group schemes*

$$\pi_1(\mathrm{Strat}^{\mathrm{rs}}((U, \overline{U})), \omega) \to \pi_1(\mathrm{Strat}^{\mathrm{rs}}((X, \overline{X})), \omega|_{\mathrm{Strat}^{\mathrm{rs}}((X, \overline{X}))})$$

induced by ρ_U is faithfully flat.

(e) *If $\mathrm{codim}_X(X \setminus U) \geq 2$, then the restriction functor ρ_U is an equivalence.* □

PROOF. If $\eta_1, \ldots, \eta_n$ are the generic points of the components of $\overline{X} \setminus X$, and $\overline{U}_i$ an open neighborhood of η_i, such that $\eta_j \in \overline{U}_i$ if and only if $j = i$, then we can take $\overline{U} := \bigcup_i \overline{U}_i \cup U$¿ This proves (a).

The restriction functor ρ_U is fully faithful, since $\mathrm{Strat}^{\mathrm{rs}}((X, \overline{X}))$ is a full subcategory of $\mathrm{Strat}(X)$, and $\mathrm{Strat}^{\mathrm{rs}}((U, \overline{(U)}))$ is a full subcategory of $\mathrm{Strat}(U)$, and ρ_U is the restriction of the restriction functor $\mathrm{Strat}(X) \to \mathrm{Strat}(U)$, which is fully faithful by Proposition 3.1.6. Hence (b) holds.

By Proposition 3.2.14, if $\iota : \mathrm{Strat}^{\mathrm{rs}}((X, \overline{X})) \hookrightarrow \mathrm{Strat}(X)$ denotes the inclusion, then ι induces an equivalence $\langle E \rangle_{\otimes} \xrightarrow{\sim} \langle \iota(E) \rangle_{\otimes}$, and similarly for U. Since by Proposition 3.1.6 restriction to U induces an equivalence $\langle \iota(E) \rangle_{\otimes} \to \langle \iota(E)|_U \rangle_{\otimes}$, statement (c) follows.

Claim (d) directly follows from Proposition C.2.3 and (c).

If $U \subseteq X$ has codimension ≥ 2, then again by Proposition 3.1.6, the functor $\mathrm{Strat}(X) \to \mathrm{Strat}(U)$ is an equivalence, so all we have to show is that a stratified bundle E on X is $(X, \overline{X})$-regular singular, if $E|_U$ is $(U, \overline{U})$-regular singular. But this is precisely Proposition 3.2.11, so (e) is also true. ■

Next, we study the functorial behavior of $\mathrm{Strat}^{\mathrm{rs}}((X,\overline{X}))$:

3.2.17 Proposition. *Let $(X,\overline{X})$, $(Y,\overline{Y})$ be good partial compactifications and $\bar{f}:\overline{X}\to\overline{Y}$ a k-morphism, such that $\bar{f}(X)\subseteq Y$. Write $f:=\bar{f}|_Y$. Then the following are true:*

(a) *For every $(X,\overline{X})$-regular singular stratified bundle on X, the stratified bundle f^*E is $(Y,\overline{Y})$-regular singular.*

(b) *If f is finite étale and* tamely ramified *(Definition B.1.1) with respect to $\overline{X}\setminus X$, then f_*E is $(X,\overline{X})$-regular singular for every $(Y,\overline{Y})$-regular singular bundle E on Y.* □

PROOF. (a) One just has to observe that $\bar{f}$ induces a morphism of the associated log-schemes, so if $\overline{E}$ is a $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log\overline{X}\setminus X)$-module extending E, then $\bar{f}^*\overline{E}$ is a $\mathcal{O}_{\overline{Y}}$-coherent $\mathscr{D}_{\overline{Y}/k}(\log\overline{Y}\setminus Y)$-module extending f^*E, see Proposition 2.3.6.

(b) Write $D_X:=\overline{X}\setminus X$ and $D_Y:=\overline{Y}\setminus Y$. If f is tamely ramified with respect to $\overline{X}\setminus X$, then the induced morphism of log-schemes is log-étale, so by Proposition 2.3.6, we get a morphism

$$\mathscr{D}_{\overline{X}/k}(\log D_X)\to\bar{f}_*\bar{f}^*\mathscr{D}_{\overline{X}/k}(\log D_X)\cong\bar{f}_*\mathscr{D}_{\overline{Y}/k}(\log D_Y).$$

If $\overline{E}$ is an $\mathcal{O}_{\overline{Y}}$-coherent $\mathscr{D}_{\overline{Y}/k}(\log\overline{Y}\setminus Y)$-module extending E, then $\bar{f}_*\overline{E}$ is an $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D_X)$-module extending f_*E, so f_*E is $(X,\overline{X})$-regular singular. ■

3.2.18 Corollary. *If $(X,\overline{X})$ is a good partial compactification and $f:Y\to X$ a finite, étale morphism, tamely ramified with respect to $\overline{X}\setminus X$, then $f_*\mathcal{O}_Y$ is $(X,\overline{X})$-regular singular.* □

PROOF. Since we may shrink $\overline{X}$ around the codimension 1 points in $\overline{X}\setminus X$ by Proposition 3.2.12, we may use Lemma B.1.9 to get into a situation where there is a $\overline{Y}$, such that $(Y,\overline{Y})$ is a good partial compactification, and such that f extends to a finite morphism $\bar{f}:\overline{Y}\to\overline{X}$, tamely ramified over $\overline{X}\setminus X$. Then we apply Proposition 3.2.17. ■

3.2.4 Topological invariance of $\mathrm{Strat}^{\mathrm{rs}}((\mathbf{X},\overline{\mathbf{X}}))$

For a k-scheme X, we write $X^{(n)}$ for the base change of X along the n-th power of the absolute frobenius of k, and by $F^{(n)}_{X/k}:X\to X^{(n)}$ the associated k-linear relative frobenius.

We have the following analog of Proposition 3.1.10:

3.2.19 Proposition. *If $(X,\overline{X})$ is a good partial compactification, then the pair $(X^{(n)},\overline{X}^{(n)})$ also is a good partial compactification, and $F^{(n)}_{X/k}$ induces an equivalence*

$$(F^{(n)}_{X/k})^*:\mathrm{Strat}^{\mathrm{rs}}((X^{(n)},\overline{X}^{(n)}))\to\mathrm{Strat}^{\mathrm{rs}}((X,\overline{X})).$$ □

PROOF. We may assume that $n = 1$, and write $F_{X/k} = F^{(1)}_{X/k}$.

Write $D := \overline{X} \setminus X$ and $D^{(1)} := \overline{X}^{(1)} \setminus X^{(1)}$. By Proposition 3.1.10 the functor $F^*_{X/k} : \mathrm{Strat}(X^{(1)}) \to \mathrm{Strat}(X)$ is an equivalence, so it suffices to show that the essential image of $\mathrm{Strat}^{\mathrm{rs}}((X^{(1)}, \overline{X}^{(1)}))$ in $\mathrm{Strat}(X)$ is $\mathrm{Strat}^{\mathrm{rs}}((X, \overline{X}))$, i.e. it suffices to show that a stratified bundle E on $X^{(1)}$ is $(X^{(1)}, \overline{X}^{(1)})$-regular singular if $F^*_{X/k}E$ is $(X, \overline{X})$-regular singular.

Assume that $F^*_{X/k}E$ is $(X, \overline{X})$-regular singular. Let E' be any torsion free coherent extension of E to $\overline{X}^{(1)}$ and $\overline{E}$ the $\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$-module generated by E'. We need to show that $\overline{E}$ is $\mathcal{O}_{\overline{X}}$-coherent. Since $F_{\overline{X}/k}$ is faithfully flat, it suffices to show that $F^*_{\overline{X}/k}\overline{E}$ is $\mathcal{O}_X$-coherent, [SGA1, Prop. VIII.1.10]. Define G to be the $\mathscr{D}_{\overline{X}/k}(\log D)$-module generated by $F^*_{\overline{X}/k}E'$. Then G is $\mathcal{O}_{\overline{X}}$-coherent by assumption, so the proof is complete if we can show that $G = F^*_{\overline{X}/k}\overline{E}$.

Let $j : X \hookrightarrow \overline{X}$ be the inclusion. The (quasi-coherent) $\mathcal{O}_{\overline{X}^{(1)}}$-module $j^{(1)}_* E$ naturally carries a $\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$-action, and similarly for $j_* F^*_{X/k}E$. Then we can describe $F^*_{\overline{X}/k}\overline{E}$ as the image of the (pulled-back) evaluation morphism

$$F^*_{\overline{X}/k}\left(\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)}) \otimes_{\mathcal{O}_{\overline{X}^{(1)}}} E'\right) \to F^*_{\overline{X}/k} j^{(1)}_* E,$$

and G as the image of the evaluation morphism

$$\mathscr{D}_{\overline{X}/k}(\log D) \otimes_{\mathcal{O}_{\overline{X}}} F^*_{\overline{X}/k}E' \to j_* F^*_{X/k}E = F^*_{\overline{X}/k} j^{(1)}_* E.$$

These morphisms fit in a commutative diagram (writing $F = F_{\overline{X}/k}$ for legibility):

$$\begin{tikzcd}
F^*\left(\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)}) \otimes_{\mathcal{O}_{\overline{X}^{(1)}}} E'\right) \arrow[r] \arrow[d, equal] & F^* j^{(1)}_* E \arrow[ddd, equal] \\
F^*\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)}) \otimes_{F^{-1}\mathcal{O}_{\overline{X}^{(1)}}} F^{-1}E' & \\
\mathscr{D}_{\overline{X}/k}(\log D^{(1)}) \otimes_{F^{-1}\mathcal{O}_{\overline{X}^{(1)}}} F^{-1}E' \arrow[u, "\gamma\otimes\mathrm{id}"] \arrow[d, equal] & \\
\mathscr{D}_{\overline{X}/k}(\log D) \otimes_{\mathcal{O}_{\overline{X}}} F^*E' \arrow[r] & j_* F^* E,
\end{tikzcd}$$

where $\gamma : \mathscr{D}_{\overline{X}/k}(\log D) \to F^*_{\overline{X}/k}\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$ is the morphism from Proposition 2.3.6. It follows that $G = F^*_{\overline{X}/k}\overline{E}$, if γ is surjective.

This is a local question, so we may assume that $\overline{X} = \operatorname{Spec} A$ and $X = \operatorname{Spec} A[t_1^{-1}]$, with $t_1, \ldots, t_r$ local coordinates on A. Then $\overline{X}^{(1)} = \operatorname{Spec} A \otimes_{F_k} k$, and $t_1 \otimes 1, \ldots, t_r \otimes 1$ is a system of local coordinates for $A \otimes_{F_k} k$. The relative frobenius $F_{X/k}$ then maps $t_i \otimes 1 \mapsto t_i^p$, for $i = 1, \ldots, r$.

Writing

$$\delta_{t_i}^{(p^m)} := \frac{t_i^{p^m}}{p^m!}\partial^{p^m}/\partial t_i^{p^m},$$

we claim that

$$\gamma(\delta_{t_i}^{(p^m)}) = \delta_{t_i\otimes 1}^{(p^{m-1})}, \tag{3.2}$$

which shows that the image of γ contains all the generators of the $\mathcal{O}_{\overline{X}}$-algebra $F^*_{X/k}\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$, and thus that γ is surjective. As for the claim, it suffices to observe that for $s \geq 0$,

$$\delta_{t_i}^{(p^m)}((t_j^p)^s) = \begin{cases} 0 & i \neq j \\ \binom{sp}{p^m} & \text{otherwise,}\end{cases}$$

and finally that

$$\binom{sp}{p^m} \equiv \binom{s}{p^{m-1}} \mod p.$$

■

3.2.20 Remark. This proof does *not* show that $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-modules descend to $\mathcal{O}_{\overline{X}^{(1)}}$-coherent $\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$-modules.

Moreover, a statement analogous to Proposition 3.2.19 for $\mathcal{O}_{\overline{X}^{(1)}}$-coherent $\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$-modules is false, due to the failure of Cartier's theorem [Kat70, §5.] for $\mathscr{D}_{\overline{X}/k}(\log D)$-modules in the form of a logarithmic analog of Theorem 3.1.11 (see [Ogu94] for a discussion, and [Lor00] for a version of Cartier's theorem for log-schemes). □

3.2.21 Corollary. *With the notations from Proposition 3.2.19, if $\overline{E}$ is a locally free $\mathcal{O}_{\overline{X}^{(1)}}$-coherent $\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$-module with exponents $\alpha_1, \dots, \alpha_n \in \mathbb{Z}_p$, then $F^*_{\overline{X}/k}\overline{E}$ is a $\mathscr{D}_{\overline{X}/k}(\log D)$-module with exponents $p\alpha_1, \dots, p\alpha_n$.*

Consequently, if $E \in \mathrm{Strat}^{\mathrm{rs}}((X^{(1)}, \overline{X}^{(1)}))$, with exponents $\alpha_1, \dots, \alpha_n \in \mathbb{Z}_p/\mathbb{Z}$, then $F^*_{X/k}E$ has exponents $p\alpha_1, \dots, p\alpha_n \in \mathbb{Z}_p/\mathbb{Z}$. □

Proof. The claim follows directly from the formula (3.2) and the fact that $\delta_{t_1}^{(1)}$ acts on $F^*_{\overline{X}/k}\overline{E} = \mathcal{O}_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}^{(1)}} \overline{E}$ via $\delta_{t_1}^{(1)}(a \otimes e) = \delta_{t_1}^{(1)}(a) \otimes e$, i.e. the "first part of the stratification" is regular. ■

3.2.22 Remark. Note that multiplication by p is an automorphism of the group $\mathbb{Z}_p/\mathbb{Z}$. □

3.2.23 Theorem. *Let $(X, \overline{X})$, $(Y, \overline{Y})$ be good partial compactifications and $\bar{f} : \overline{Y} \to \overline{X}$ a universal homeomorphism such that $\bar{f}(X) \subseteq \overline{Y}$. If we write $f := \bar{f}|_X$ then f induces an equivalence*

$$f^* : \mathrm{Strat}^{\mathrm{rs}}((X, \overline{X})) \xrightarrow{\sim} \mathrm{Strat}^{\mathrm{rs}}((Y, \overline{Y})).$$

□

Proof. This is the same argument as in the proof of Theorem 3.1.11, using Proposition 3.2.19 instead of Proposition 3.1.10: There is a morphism $g : \overline{Y} \to \overline{X}^{(n)}$, such that $g(Y) \subseteq X^{(n)}$. Then $(gf)^*$ is an equivalence, so g^* is essentially surjective. Since g is faithfully flat, g^* is an equvalence, so f^* is also an equivalence. ■

3.3 Regular singular bundles in general

Since resolution of singularities is not available in positive characteristic, we unfortunately cannot define what it means for a stratified bundle on a smooth k-scheme to be regular singular by considering a compactification $\overline{X}$, such that $\overline{X} \setminus X$ is a strict normal crossings divisor. In this section we present a definition that works in general, and we generalize the results from Section 3.2 to this context.

We freely use the notations and results from Appendix A.

3.3.1 Definition. Let X/k be a smooth, separated, finite type k-scheme and E a stratified bundle on X. Let ν be a discrete valuation on $k(X)$ and $\mathcal{O}_\nu \subseteq k(X)$ its valuation ring. Then E is called

(a) *ν-regular*, if the $\mathscr{D}_{X/k} \otimes k(X)$-module $E \otimes k(X)$ comes via base change from a free $\mathcal{O}_\nu$-module of finite rank, which carries a compatible $\mathscr{D}_{\mathcal{O}_\nu/k}$-action.

(b) *ν-regular singular*, if the $\mathscr{D}_{X/k} \otimes k(X)$-module $E \otimes k(X)$ comes via base change from a free $\mathcal{O}_\nu$-module of finite rank, which carries a compatible $\mathscr{D}_{\mathcal{O}_\nu/k}(\log \mathfrak{m}_\nu)$-action, if $\mathfrak{m}_\nu$ is the maximal ideal of $\mathcal{O}_\nu$.

(c) *regular singular*, if E is ν-regular singular for all *geometric* discrete valuations on $k(X)$, i.e. for all discrete valuations coming from codimension 1 points on some normal compactification of X.

Denote by $\mathrm{Strat}^{\mathrm{rs}}(X)$ the full subcategory of $\mathrm{Strat}(X)$ with objects the regular singular stratified bundles. □

3.3.2 Remark. Now the separatedness condition on X is really necessary, since we are using Nagata's compactification theorem [Lüt93]. □

Next, we show that Definition 3.3.1 is indeed a reasonable definition, i.e. that it reduces to the usual definition of regular singularity in the presence of a good resoultion of singularities.

3.3.3 Proposition. *Let X be a smooth, separated, finite type k-scheme and E a stratified bundle on X. Then the following are equivalent:*

(a) *E is regular singular.*

(b) *E is $(X,\overline{X})$-regular singular for all good* partial *compactifications $(X,\overline{X})$ of X.*

(c) *For any normal compactification X^N of X, and any open subset $\overline{X} \subseteq X^N$ with $\operatorname{codim}_{X^N}(\overline{X}) \geq 2$, such that $X \subseteq \overline{X}$, and such that $(X,\overline{X})$ is a good partial compactification, E is $(X,\overline{X})$-regular singular.*

If X admits a good compactification $\overline{X}$, i.e. if there exists a smooth proper k-scheme $\overline{X}$, such that $X \subseteq \overline{X}$ and such that $\overline{X} \setminus X$ is a strict normal crossings divisor, then (a)-(c) *are equivalent to*

(d) *E is $(X,\overline{X})$-regular singular.* □

PROOF. Assume E is regular singular and let $(X,\overline{X})$ be a good partial compactification. The codimension 1 points $\eta_1,\dots,\eta_r$ of $\overline{X}$ lying in $\overline{X}\setminus X$ correspond to discrete valuations $\nu_1,\dots,\nu_r$, and since E is regular singular, there are open neighborhoods $\overline{U}_i$ of η_i, such that we can extend E to an $\mathscr{O}_{\overline{U}_i}$-coherent $\mathscr{D}_{\overline{U}_i/k}(\log \overline{U}_i\setminus X)$-module. In other words: $E|_{\overline{U}_i\cap X}$ is $(X\cap\overline{U}_i,\overline{U}_i)$-regular singular for every i, but then E is also $(X,\overline{X})$-regular singular by Proposition 3.2.12. Thus (a) implies (b).

Clearly (b) implies implies (c). To see that (c) implies (a), let ν be a geometric discrete valuation of $k(X)$, i.e. a discrete valuation ν, such that there exists some normal compactification X^N of X and a codimension 1 point $\eta\in X^N$, with $\nu=\nu_\eta$. Then, if (c) holds, E is ν-regular singular. This can be done for every ν, so E is regular singular.

From now on assume that X has a good compactification $\overline{X}$. Then (b) trivially implies (d), so finally assume that E is $(X,\overline{X})$-regular singular. Let ν be a geometric discrete valuation of $k(X)$, and X^N a normal compactification of X such that ν corresponds to a codimension 1 point η of X^N. If $\eta\in X$, then E is trivially ν-regular singular. If $\eta\notin X$, then we argue as follows: There is a rational map $G: X^N \dashrightarrow \overline{X}$, which is a morphism over some open subset $\overline{U}\subseteq X^N$ which contains all codimension 1 points of X^N. Shrinking $\overline{U}$ around η, we may assume that $\overline{U}$ is smooth over k, $X\subseteq\overline{U}$, and that η is the only generic point of $X^N\setminus X$ contained in $\overline{U}$. The pair $(X,\overline{U})$ is a good partial compactification. It then suffices to check that E is $(X,\overline{U})$-regular singular. But if $\overline{E}$ is a locally free $\mathscr{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log\overline{X}\setminus X)$-module extending E, then $G^*\overline{E}$ is a $\mathscr{O}_{\overline{U}}$-coherent $\mathscr{D}_{\overline{U}/k}(\log\overline{U}\setminus X)$-module extending E, by Proposition 3.2.17, so E is $(X,\overline{U})$-regular singular. ■

The category $\mathrm{Strat}^{\mathrm{rs}}(X)$ of regular singular bundles still has many of the good properties of $\mathrm{Strat}^{\mathrm{rs}}((X,\overline{X}))$ for a good partial compactification $(X,\overline{X})$:

3.3.4 Proposition. *Let X be a smooth, separated, finite type k-scheme, and let $(X,\overline{X}_1)$ and $(X,\overline{X}_2)$ be good partial compactifications.*

(a) *If $(X,\overline{X}_1)\sim(X,\overline{X}_2)$ (see Definition A.3), then*

$$\mathrm{Strat}^{\mathrm{rs}}((X,\overline{X}_1))=\mathrm{Strat}^{\mathrm{rs}}((X,\overline{X}_2))$$

as full tannakian subcategories of $\mathrm{Strat}(X)$.

(b) *If $(X,\overline{X}_1)\leq(X,\overline{X}_2)$, then*

$$\mathrm{Strat}^{\mathrm{rs}}((X,\overline{X}_2))\subseteq\mathrm{Strat}^{\mathrm{rs}}((X,\overline{X})_1)$$

as full tannakian subcategories of $\mathrm{Strat}(X)$.

(c) *If we write* $\mathrm{Strat}^D(X):=\mathrm{Strat}^{\mathrm{rs}}((X,\overline{X}))$ *for $D\in\mathrm{PC}(X)$ an equivalence class of partial compactifications, and $(X,\overline{X})\in D$, then the set of tannakian subcategories* $\{\mathrm{Strat}^D(X)|D\in\mathrm{PC}(X)\}$ *is an inductive system with respect to inclusion.*

(d) $\mathrm{Strat}^{\mathrm{rs}}(X)=\bigcap_{D\in\mathrm{PC}(X)}\mathrm{Strat}^D(X)$.

(e) $\mathrm{Strat}^{\mathrm{rs}}(X)$ *is a tannakian subcategory of* $\mathrm{Strat}(X)$.

(f) *In particular, if* $\iota : \mathrm{Strat}^{\mathrm{rs}}(X) \hookrightarrow \mathrm{Strat}(X)$ *is the inclusion, then for every* $E \in \mathrm{Strat}^{\mathrm{rs}}(X)$, ι *restricts to an equivalence* $\langle E \rangle_{\otimes} \xrightarrow{\sim} \langle \iota(E) \rangle_{\otimes}$. □

3.3.5 Remark. Since we are dealing with full subcategories (even *strictly full* subcategories) of the fixed ambient category $\mathrm{Strat}(X)$, there is no difficulty in defining a directed system of such subcategories or their intersection. We may be ignoring set theoretic problems though. □

PROOF (OF PROPOSITION 3.3.4). The points (a) to (d) are clear from the definitions, Corollary A.5 and Proposition 3.3.3.

Claim (e) and (f) follow easily using the characterization from Proposition 3.3.3, (b) together with Proposition 3.2.14: If $E \in \mathrm{Strat}^{\mathrm{rs}}(X)$ is a regular singular stratified bundle, then E is $(X, \overline{X})$-regular singular for all good partial compactifications $(X, \overline{X})$ of X, so by Proposition 3.2.14 every subquotient in $\mathrm{Strat}(X)$ is $(X, \overline{X})$-regular singular for every good partial compactification $(X, \overline{X})$, so every such subquotient lies in $\mathrm{Strat}^{\mathrm{rs}}(X)$. The same argument works for tensor products and internal Homs. This proves (e) and (f). ■

3.3.6 Proposition. *Let* $f : Y \to X$ *be a dominant* k*-morphism of smooth, separated, finite type* k*-schemes. If* $E \in \mathrm{Strat}^{\mathrm{rs}}(X)$ *then* $f^*E \in \mathrm{Strat}^{\mathrm{rs}}(Y)$. *If* f *is finite étale and tamely ramified (Definition B.2.1), then* $f_*E' \in \mathrm{Strat}^{\mathrm{rs}}(X)$ *for* $E' \in \mathrm{Strat}^{\mathrm{rs}}(Y)$. □

PROOF. We first prove the statement about f^*. Let ν be a geometric discrete valuation on $k(Y)$. By Proposition A.7 we find good partial compactifications $(Y, \overline{Y})$, $(X, \overline{X})$ and a morphism $\bar{f} : \overline{Y} \to \overline{X}$, such that $\bar{f}|_Y = f$, and such that $(Y, \overline{Y})$ realizes the valuation ν. Applying Proposition 3.2.17 shows that f^*E is ν-regular singular. Since we can do this for every geometric discrete valuation ν, it follows that f^*E is regular singular.

For the statement about f_*, assume that f is finite étale and tamely ramified. Then f_*E' is a stratified bundle on X. To check that it is regular singular, it suffices to consider good partial compactifications $(X, \overline{X})$ such that $\overline{X} \setminus X$ is smooth, say with generic point η. Using Lemma B.1.9, we reduce to the situation where $(Y, \overline{Y})$ is a good partial compactification, $\bar{f} : \overline{Y} \to \overline{X}$ a finite sujective morphism, tamely ramified over $\overline{X} \setminus X$ and $\bar{f}|_Y = f$. Then we can apply Proposition 3.2.17 to show that f_*E' is $(X, \overline{X})$-regular singular, and since this works for every such good partial compactification $(X, \overline{X})$ we are done. ■

3.3.7 Remark. Of course one would expect f^*E to be regular singular, even if f is not dominant, e.g. if $f : C \to X$ is a closed immersion of a curve into X. At the moment of writing, the author does not know how to prove this. Note that also in characteristic 0 this is the hardest part of proving that a stratified bundle is regular singular if and only if it is regular singular on all curves. A purely algebraic proof has been given only fairly recently; see [And07]. □

Just like in Proposition 3.1.6 and Proposition 3.2.16, if $U \subseteq X$ is an open subset, we can analyze the restriction functor $\rho_U : \mathrm{Strat}^{\mathrm{rs}}(X) \to \mathrm{Strat}^{\mathrm{rs}}(U)$.

3.3.8 Proposition. *Let* X *be a smooth, separated, finite type* k*-scheme and* U *a dense open subset. Then the the following statements are true.*

(a) *For* $E \in \mathrm{Strat}^{\mathrm{rs}}(X)$*, the restriction functor* $\langle E\rangle_{\otimes} \to \langle E|_U\rangle_{\otimes}$ *is an equivalence.*

(b) *The restriction functor* $\rho_U : \mathrm{Strat}^{\mathrm{rs}}(X) \to \mathrm{Strat}^{\mathrm{rs}}(U)$ *is fully faithful.*

(c) *If* $\omega : \mathrm{Strat}^{\mathrm{rs}}(U) \to \mathrm{Vectf}_k$ *is a fiber functor, then the induced morphism of* k*-group schemes*

$$\pi_1(\mathrm{Strat}^{\mathrm{rs}}(U), \omega) \to \pi_1(\mathrm{Strat}^{\mathrm{rs}}(X), \omega|_{\mathrm{Strat}^{\mathrm{rs}}(X)})$$

is faithfully flat.

(d) *If* $\mathrm{codim}_X(X \setminus U) \geq 2$*, then* ρ_U *is an equivalence.* □

PROOF. The arguments are almost the same as in Proposition 3.2.16.

(a) follows from Proposition 3.3.4 and Proposition 3.1.6.

(b) is true, as $\mathrm{Strat}^{\mathrm{rs}}(X)$ and $\mathrm{Strat}^{\mathrm{rs}}(U)$ are full subcategories of $\mathrm{Strat}(X)$, resp. $\mathrm{Strat}(U)$, and, according to Proposition 3.1.6, the restriction functor $\mathrm{Strat}(X) \to \mathrm{Strat}(U)$ is fully faithful.

(c) again follows from Proposition C.2.3 and (a).

Finally, to show that (d) is true, we have to show that ρ_U is essentially surjective. We know that $\mathrm{Strat}(X) \to \mathrm{Strat}(U)$ is essentially surjective, so we have to show that if E is a stratified bundle on X, such that $E|_U$ is regular singular, then E is regular singular. But this follows from Proposition 3.3.3. ■

3.3.9 Example. Every stratified line bundle on X is regular singular, because it is $(X, \overline{X})$-regular singular for every good partial compactification $(X, \overline{X})$, see Example 3.2.4. □

3.3.1 Exponents

In this section we extend the notion of exponents to general regular singular stratified bundles. This is fairly straightforward: Let E be a regular singular bundle on X. We know what exponents of E with respect to a partial compactification are, and it is easy to show that exponents with respect to two equivalent partial compactifications agree. Thus for every codimension 1 point in some normal compactification X^N of X we get a well-defined set of exponents, and we can pass to the union over the directed set of equivalence classes of partial compacitifications, to get a set of exponents of E.

3.3.10 Proposition. *Let* X *be a smooth, connected, separated, finite type* k*-scheme. Let* $(X, \overline{X}_1)$ *and* $(X, \overline{X}_2)$ *be two good partial compactifications, such that there are codimension* 1 *points* $\eta_1 \in \overline{X}_1$*,* $\eta_2 \in \overline{X}_2$*, such that* $\nu_{\eta_1} = \nu_{\eta_2}$*. Then the set of exponents of* E *along* η_1 *and the set of exponents exponents of* E *along* η_2 *agree.*

In particular, if $(X, \overline{X}_1)$ *and* $(X, \overline{X}_2)$ *are good partial compactifications, then*

(a) $\mathrm{Exp}_{(X,\overline{X}_1)}(E) = \mathrm{Exp}_{(X,\overline{X}_2)}(E)$ *if* $(X, \overline{X}_1) \sim (X, \overline{X}_2)$*,*

(b) $\mathrm{Exp}_{(X,\overline{X}_1)}(E) \subseteq \mathrm{Exp}_{(X,\overline{X}_2)}(E)$ *if* $(X, \overline{X}_1) \leq (X, \overline{X}_2)$*.*

For the definition of $\mathrm{Exp}_{(X,\overline{X}_i)}(E)$*, see Definition 3.2.6.* □

PROOF. Let X_2^N be a normal compactification of $\overline{X}_2$. Since X_2^N is proper over k, there is an open subset $\overline{U} \subseteq \overline{X}_1$, such that $\operatorname{codim}_{\overline{X}_1}(\overline{X}_1 \setminus \overline{U}) \geq 2$, and a morphism $f : \overline{U} \to X_2^N$ restricting to the identity on X. We show that $f(\eta_1) = \eta_2$ and that there is an open neighborhood $\overline{U}_{\eta_1}$ of $\eta_1 \in \overline{U}$, such that $f|_{\overline{U}_{\eta_1}}$ is an isomorphism onto its image.

Indeed, if $f(\eta_1) \neq \eta_2$, then there would be two ways different dotted arrows in the following diagram:

$$\begin{tikzcd}
\operatorname{Spec} k(X) \arrow[r] \arrow[d] & X^N \arrow[d] \\
\operatorname{Spec} \mathcal{O}_{\nu_1} \arrow[r] \arrow[ur, dotted] & \operatorname{Spec}(k)
\end{tikzcd}$$

which cannot happen, since X^N is separated over k. This implies that we can find an open set $\overline{U}_{\eta_1}$ containing X and η_1, such that $f(\overline{U}_{\eta_1}) \subseteq \overline{X}_2$. We are now in the following situation

$$\begin{tikzcd}
\overline{X}_1 & & \\
\overline{U} \arrow[u, hook] \arrow[rr, "f"] & & X_2^N \\
\eta_1 \in \overline{U}_{\eta_1} \arrow[u, hook] \arrow[rr] & & \overline{X}_2 \ni \eta_2 \arrow[u, hook] \\
& X \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Finally, note that since $\overline{U}_{\eta_1}$ and $\overline{X}_2$ are both regular schemes of the same dimension, f is flat at η_1. This means there is an open subset $\overline{U}_2$ of $\overline{X}_2$ containing X and η_2, such that f restricted to the open set $f^{-1}(\overline{U}_2) \subseteq \overline{U}_{\eta_1}$ is flat. But since f is birational, this means that $f|_{f^{-1}(\overline{U}_2)}$ is an isomorphism onto its image. ∎

3.3.11 Definition. Let X be a smooth, separated, finite type k-scheme, $E \in \operatorname{Strat}^{\mathrm{rs}}(X)$ a regular singular stratified bundle on X, and ν a discrete geometric valuation on $k(X)$. We define the *set of exponents* $\operatorname{Exp}_\nu(E)$ *of* E *along* ν to be the exponents of E along any divisor in a good partial compactification $(X, \overline{X})$ with valuation ν. Proposition 3.3.10 shows that this is a well-defined notion.

We define the *set of exponents of* E to be

$$\operatorname{Exp}(E) := \bigcup_{(X,\overline{X})} \operatorname{Exp}_{(X,\overline{X})}(E) \subseteq \mathbb{Z}_p/\mathbb{Z}$$

where the union runs over all good partial compactifications of X. □

3.3.12 Proposition. *In the situation of Definition 3.3.11,*

$$\operatorname{Exp}(E) = \bigcup_{D \in \operatorname{PC}(X)} \operatorname{Exp}_D(E) = \bigcup_{\nu \in \operatorname{DVal}^{\mathrm{geom}}(X)} \operatorname{Exp}_\nu(E),$$

where for $D \in \mathrm{PC}(X)$, we write $\mathrm{Exp}_D(E) = \mathrm{Exp}_{(X,\overline{X})}(E)$ *for* $(X, \overline{X}) \in D$. □

Proof. This is just a reformulation of the definition of $\mathrm{Exp}(E)$ and Proposition 3.3.10. ■

3.3.13 Question. *If X admits a good compactification and if E a regular singular stratified bundle on X, then it is easy to show that the subgroup of $\mathbb{Z}_p/\mathbb{Z}$ generated by the exponents of E is finitely generated (if P is a point lying on multiple components of the boundary divisor, then the exponents of E along codimension 1 points in the blowing-up of X in P are a linear combination of the exponents along the components on which P lies).*

Is this also true if one does not assume resolution of singularities? We will discuss this question further (although unfortunately not conclusively) in Section 3.4. □

3.3.2 Topological invariance of $\mathrm{Strat}^{\mathrm{rs}}(\mathbf{X})$

Just as for $(X, \overline{X})$-regular singular bundles (Section 3.2.4), and for the category of tamely ramified covers of X, we can show that the category $\mathrm{Strat}^{\mathrm{rs}}(X)$ is a "topological invariant" of X.

3.3.14 Theorem. *Let $f : Y \to X$ be a universal homeomorphism of smooth, separated, finite type k-schemes. Then f induces an equivalence*

$$f^* : \mathrm{Strat}^{\mathrm{rs}}(X) \xrightarrow{\sim} \mathrm{Strat}^{\mathrm{rs}}(Y).$$ □

Proof. Since $f^* : \mathrm{Strat}(X) \to \mathrm{Strat}(Y)$ is an equivalence by Theorem 3.1.11, we just need to check that if f^*E is regular singular for $E \in \mathrm{Strat}(X)$, then E is regular singular.

For this we need to show that for every good partial compactification $(X, \overline{X})$, such that $\overline{X} \setminus X$ is smooth with generic point η, there exists an open neighborhood $\overline{U}$ of η, an open subset $V \subseteq Y$, and a good partial compactification $(V, \overline{V})$, such that f induces a universal homeomorphism

$$\bar{g} : \overline{V} \to \overline{U}$$

with $g(V) \subseteq U := \overline{U} \cap X$. Then we can apply Theorem 3.2.23 to finish.

Let $\overline{U} = \operatorname{Spec} A$ be an affine open neighborhood of η, such that $U := \overline{U} \cap X = \operatorname{Spec} A[x_1^{-1}]$ for some regular element $x_1 \in A$. Because f is finite, $V := f^{-1}(U)$ is also affine, say $V = \operatorname{Spec} B$, and $f|_V$ is a universal homeomorphism. Let $\overline{B}$ be the integral closure of A in B. Then $f_{\overline{B}} : \operatorname{Spec} \overline{B} \to \operatorname{Spec} A$ is a finite morphism. By construction, $\overline{V} := \operatorname{Spec} \overline{B}$ is normal, but perhaps not smooth. Nevertheless, there is precisely one codimension 1 point in $\operatorname{Spec} \overline{B}$ lying over η (because $k(X) \hookrightarrow k(Y)$ is purely inseparable), and we may shrink $\overline{U}$ around η. There is an open neighborhood $\overline{U}' \subseteq \overline{U}$ of η, such that $\overline{V}' := f_{\overline{B}}^{-1}(\overline{U}')$ is smooth, and $\overline{V}' \to \overline{U}'$ is precisely the universal homeomorphism we were looking for. ■

3.4 Regular singularities with respect to curves and related notions

This section is inspired by [KS10]. Besides our definition of regular singular stratified bundles (Definition 3.3.1) there are numerous other possible definitions that turn out to be equivalent in characteristic 0. Unfortunately we cannot prove similarly strong comparison theorems in positive characteristic (yet), but nevertheless, we can prove a few things under additional hypotheses on k.

We continue to denote by k an algebraically closed base field of characteristic $p > 0$.

3.4.1 Definition. Let X be a smooth, separated, finite type k-scheme. A stratified bundle E on X is said to be

(a) *curve-r.s.* if for every morphism $\phi : C \to X$ with C a regular curve over k, the stratified bundle ϕ^*E is $(C, \overline{C})$-regular singular, where $\overline{C}$ is the regular compactification of C.

(b) *weakly curve-r.s.* if the stratified bundle ϕ^*E is $(C, C \cup \{P\})$-regular singular, for every morphism $\phi : C \to X$ with C a regular curve, and for every $P \in \overline{C} \setminus C$, with the property that there exists a normal compactification X_P^N, such that for the extension $\phi' : C \cup \{P\} \to X_P^N$, we have $\phi'(P) \in (((X_P^N) \setminus X)_{\mathrm{red}})_{\mathrm{reg}}$.

(c) *divisor-r.s.* if E is ν-regular singular for every geometric discrete valuation (Definition A.1) on $k(X)$.

(d) *weakly divisor-r.s.*, if there exists a normal compactification X^N, such that if $\eta_1, \ldots, \eta_r \in X^N \setminus X$ are the codimension 1 points of $X^N \setminus X$, then E is η_i-regular singular for $i = 1, \ldots, r$.

Clearly, a stratified bundle E is weakly divisor-r.s. if it is divisor-r.s., and weakly curve-r.s. if it is curve-r.s. □

3.4.2 Remark. Our ultimate goal is to prove that the notions "curve-r.s." and "divisor-r.s." and "weakly divisor-r.s." are equivalent on smooth, separated, finite type k-schemes. This is true in characteristic 0, see e.g. [Del70, Prop. 4.4]. For now we have to content ourselves with the following results. The notion "weakly curve-r.s." should be understood as an auxiliary definition. □

Before we come to the main result of this section, we make an immediate observation:

3.4.3 Proposition. *If X is a smooth* curve*, i.e. a smooth, separated, finite type k-scheme of dimension 1, and $\overline{X}$ the smooth compactification of X, then the four notions of regular singularity from Definition 3.4.1 agree, and are equivalent to E being $(X, \overline{X})$-regular singular.* □

PROOF. This is trivial. ■

3.4.4 Theorem. *Let X be a smooth, separated, finite type k-scheme of dimension ≥ 1. For a stratified bundle E on X, consider the following diagram:*

$$
\begin{tikzcd}
\textit{E is curve-r.s.} \arrow[d, Rightarrow, "(\alpha)"'] & \textit{E is weakly divisor-r.s.} \\
\textit{E is weakly curve-r.s.} & \textit{E is divisor-r.s.} \arrow[l, Rightarrow, "(\beta)"] \arrow[u, Rightarrow, "(\gamma)"] \arrow[ul, Rightarrow, dashed, "(\delta)"]
\end{tikzcd}
\tag{3.3}
$$

The following statements are true:

(a) *The solid implications from diagram* (3.3) *hold unconditionally.*

(b) *If k is uncountable, then the arrow (β) is an equivalence.*

(c) *If X admits a smooth compactification $\overline{X}$ with $\overline{X} \setminus X$ a strict normal crossings divisor, then*

- (Proposition 3.3.3) *E is divisor-r.s. if and only if E is $(X, \overline{X})$-regular singular.*
- *The implication (δ) holds.*
- *If k is uncountable, then (α), (β) and (δ) are equivalences.*

(d) *If E has finite monodromy, i.e. if the k-group scheme $\pi_1(\langle E \rangle_{\otimes}, \omega)$ is finite for some fiber functor $\omega : \langle E \rangle_{\otimes} \to \mathrm{Vectf}_k$, then (α), (β), and (δ) are equivalences.* □

3.4.5 Corollary. *If X is a smooth, separated, finite type k-scheme, and E a stratified bundle on X such that $\phi^* E$ is regular singular for every k'-morphism $\phi : C \to X_{k'}$, for k' a countable, algebraically closed extension of k and C a regular k'-curve, then E is regular singular.* □

PROOF. We need to show that E is ν-regular singular for every geometric discrete valuation ν of X, if the condition of the corollary is satisfied. Let ν be a geometric discrete valuation of X, and $(X, \overline{X})$ a good partial compactification of X realizing ν. As in the proof of Theorem 3.4.4, (b) below, we reduce to $\overline{X}$ affine and E free, so that showing that E is $(X, \overline{X})$-regular singular boils down to showing that the pole order of a certain countable set of functions in $k(X)$ has a common bound. This is independent of the coefficients, so we may base change to a field $K \supseteq k$, K algebraically closed and uncountable (e.g. $K := \overline{k((t))}$). Then we can apply the theorem, to see that E_K is regular singular if it is regular singular along all regular K-curves. But every such curve is defined over a countable, algebraically closed subfield k' of K, so the corollary follows. ■

3.4.6 Remark. • Theorem 3.4.4, (d) will follow from the results of Chapter 4. In fact, (δ) is an equivalence by Theorem 4.6.5.

Assume that E is weakly curve-r.s. and let $(X, \overline{X})$ be a good partial compactification. We want to show that E is $(X, \overline{X})$-regular singular, and for this we may assume that $\overline{X} \setminus X$ is regular. Let $\bar{\phi} : \overline{C} \to \overline{X}$ be a morphism with $\overline{C}$ a (possibly affine) regular curve. Write $C := \bar{\phi}^{-1}(X)$ and $\phi := \bar{\phi}|_C$. Then by the definition of weakly curve-r.s., $\phi^* E$ is $(C, \overline{C})$-regular singular. Thus Theorem 4.5.1 implies that E is $(X, \overline{X})$-regular singular. We can do this for all partial compactifications $(X, \overline{X})$ with $\overline{X} \setminus X$ regular, so we see that the arrow (β) is an equivalence, which implies that (α) is an equivalence as well.

- The uncountability condition in (b) is annoying and can perhaps be removed.
- The problem in showing that "divisor-r.s." implies "curve-r.s." is that "divisor-r.s." does not give a good condition for a stratified bundle to be regular singular in a *singular point* P in $X^N \setminus X$ for a normal compactification X^N. So if $\phi : C \to X$ is a morphism from a regular curve to X, and $Q \in \overline{C}$ a point of the regular compactification of C, such that $\phi(Q) = P$, then it is unclear how to connect the (ir)regularity of E at P to the (ir)regularity of ϕ^*E at Q. This is why we have to resort to the notion of "weakly curve-r.s.".
- Note that an analog of [Del70, Prop. 4.6] in our characteristic $p > 0$ context fails: *loc. cit.* says that in characteristic 0, a stratified bundle E on a smooth, separated, finite type k-scheme is regular singular if and only if f^*E is regular singular for some dominant morphism $f : Y \to X$, with Y smooth, separated, and of finite type over k. If $\operatorname{char} k = p > 0$, and $\mathbb{A}^1_k = \operatorname{Spec} k[T]$, then we can take $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ to be the Artin-Schreier covering given by $T \mapsto T^p - T$. This is a finite étale morphism so $f_*\mathcal{O}_{\mathbb{A}^1_k}$ naturally carries the structure of a stratified bundle by Proposition 3.1.2, and $f^*f_*\mathcal{O}_{\mathbb{A}^1_k}$ is the trivial stratified bundle of rank p, because the covering $\mathbb{A}^1_k \times_f \mathbb{A}^1_k \to \mathbb{A}^1_k$ is the trivial covering, see e.g. Lemma 4.3.1. But $f_*\mathcal{O}_{\mathbb{A}^1_k}$ is *not* regular singular: We saw in Example 1.4.4 that formally locally around $\infty \in \mathbb{P}^1_k$ (i.e. on $K := k(\!(T^{-1})\!)$), $f_*\mathcal{O}_{\mathbb{A}^1_k}$ contains a nontrivial extension of $\mathcal{O}_K$ by itself, but $f_*\mathcal{O}_{\mathbb{A}^1_k}$ is regular singular if and only if it is a direct sum of rank 1 objects formally locally around ∞. □

PROOF (OF THEOREM 3.4.4 (A)). Clearly the implications (α) and (γ) hold. It remains to show that a divisor-r.s bundle E on X is always weakly curve-r.s. For this let $\phi : C \to X$ be a morphism of a regular k-curve to X, $\overline{C}$ the regular compactification of C and $P \in \overline{C} \setminus C$ a closed point such that there exists a normal compactification $\overline{X}^N_P$, such that the canonical extension $\bar{\phi} : \overline{C} \to X^N_P$ has the property that $\bar{\phi}(P) \in ((X^N_P \setminus X)_{\mathrm{red}})_{\mathrm{reg}}$. But this means that $\bar{\phi}(P)$ is a regular point of precisely one component of $(X^N_P \setminus X)_{\mathrm{red}}$. In other words, there is an open subset $\overline{X} \subseteq X^N_P$, such that $P \in \overline{X}$, and such that $(X, \overline{X})$ is a good compactification. Let $C' := C \cup \{P\}$, then $\bar{\phi}$ restricts to a morphism $\phi' : C' \to \overline{X}$, such that $\phi'|_C = \phi$. This means ϕ' gives a morphism of the associated log-schemes. Finally, if $\overline{E}$ is a $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$-module extending E, then $\phi'^*\overline{E}$ is a $\mathcal{O}_{C'}$-coherent $\mathscr{D}_{C'/k}(\log P)$-module extending ϕ^*E, which proves that ϕ^*E is (C, C')-regular singular. This shows that E is weakly curve-r.s.. ■

For the proof of Theorem 3.4.4 (b), we need the following observation:

3.4.7 Lemma. *Let S be a regular noetherian scheme, $X \to S$ a smooth morphism, with $X = \operatorname{Spec} A$ affine, and $U = \operatorname{Spec} A[t^{-1}]$, with $t \in A \setminus (A^\times \cup \{0\})$. Assume that the closed subscheme $D := V(t) \subseteq X$ is smooth over S. If $g \in A[t^{-1}]$, then the set*

$$\mathrm{Pol}_{\leq n}(g) := \{s \in S |\ g|_{U_s} \in \Gamma(U_s, \mathcal{O}_{U_s}) \text{ has pole order } \leq n \text{ along } D_s\}$$

is a constructible subset of S. □

PROOF. Note that since $D \to S$ is smooth, $D_s \subseteq X_s$ is a smooth divisor for every $s \in S$, so it makes sense to talk about the pole order of $g|_{U_s}$ along D_s.

Since $\mathrm{Pol}_{\leq n}(g) = \mathrm{Pol}_{\leq 0}(t^n g)$, it suffices to show that $\mathrm{Pol}_{\leq 0}(g)$ is constructible.

The element g defines a commutative diagram of S-schemes

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "g"] & X \arrow[d, "g"] \\ \mathbb{A}^1_S \arrow[r, hook] & \mathbb{P}^1_S. \end{tikzcd}$$

The image $g(X) \subseteq \mathbb{P}^1_S$ is a constructible set, so $g(X) \cap (\{\infty\} \times S)$ is a constructible subset of $\mathbb{P}^1_S$. If $\mathrm{pr} : \mathbb{P}^1_S \to S$ is the structure morphism of $\mathbb{P}^1_S$, then $\mathrm{pr}(g(X) \cap (\{\infty\} \times S))$ is a constructible subset of S. Finally note that $S \setminus \mathrm{Pol}_{\leq 0}(g) = \mathrm{pr}((g(X) \cap (\{\infty\} \times S))$. ■

PROOF (OF THEOREM 3.4.4,(B)). Let k be an algebraically closed field of characteristic $p > 0$, and assume that k is uncountable. We immediately reduce to the case where X is connected and $\dim X \geq 2$. Assume that E is weakly curve-r.s. and let ν be a divisorial valuation on $k(X)$. By Proposition 3.3.3, it is enough to check that E is $(X, \overline{X})$-regular singular for some good partial compactification $(X, \overline{X})$-realizing ν. Then ν corresponds to a generic point $\eta \in D := \overline{X} \setminus X$. By shrinking $\overline{X}$ around η, we may assume that $\overline{X} \setminus X$ is a smooth divisor with generic point η. Shrinking $\overline{X}$ further, we may assume that

- $\overline{X} = \operatorname{Spec} A$ is affine,
- there exist coordinates $x_1, \dots, x_n \in A$ such that $D = V(x_1)$.
- E corresponds to a free $A[x_1^{-1}]$-module, say with basis $e_1, \dots, e_r$.

If we write $\delta_{x_1}^{(m)} := \frac{x_1^m}{m!} \partial^m / \partial x_1^m \in \mathscr{D}_{X/k}$, then for $f \in A$ we can also write

$$\delta_{x_1}^{(m)}(f e_i) = \sum_{j=1}^{r} b_{ij}^{(m)}(f) e_j, \text{ with } b_{ij}^{(m)}(f) \in A[x_1^{-1}].$$

To show that E is regular singular, it suffices to show that the $b_{ij}^{(m)}(f)$ have bounded pole order along x_1, because then the $\mathscr{D}_{\overline{X}/k}(\log D)$-module generated by $\sum_{i=1}^r e_i A$ is contained in $x_1^{-N} \bigoplus_{i=1}^r e_i A$, and thus finitely generated over A, if N is the bound.

Since A is a finite type k-algebra, say $A = k[y_1, \dots, y_d]/I$, it suffices to show that the pole order of $b_{ij}^{(m)}(\bar{y}_c^h)$ has a common upper bound, for i, j, m, $h = 1, \dots, d$, $h \geq 0$, and $\bar{y}_c$ the image of y_c in A. Note that these are countably many elements of $A[x_1^{-1}]$.

Define $S := \mathbb{A}_k^{n-1} = \operatorname{Spec} k[x_2, \dots, x_n]$. We get a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] & \overline{X} \arrow[r, "\text{étale}"] \arrow[dr, "\text{smooth}"'] & \mathbb{A}^1_S \arrow[d] \\ & & S \end{tikzcd}$$

Now consider the constructible sets $\mathrm{Pol}_{\leq N}(b_{ij}^{(m)}(\bar{y}_c^h)) \subseteq S$ from Lemma 3.4.7 and define

$$\mathbf{P}_{\leq N} := \bigcap_{i,j,m,c,h} \overline{\mathrm{Pol}_{\leq N}(b_{ij}^{(m)}(\bar{y}_c^h))}.$$

This is a closed subset of S. Now since for every *closed* point $s \in S$ the fiber X_s is a regular curve over k, we see that by assumption $E|_{X_s}$ is $(X_s, \overline{X}_s)$-regular singular. But this means that there is some $N_s \geq 0$, such that $s \in \mathbf{P}_{\leq N_s}$. In other words,

$$\bigcup_{N \geq 0} \mathbf{P}_{\leq N}(k) = S(k).$$

Since k is uncountable and since the $\mathbf{P}_{\leq N}$ are closed subsets of S, this means there exists some $N_0 \geq 0$, such that $\mathbf{P}_{\leq N_0} = S$ (see e.g. [Liu02, Ex. 2.5.10]). The definition of $\mathbf{P}_{\leq N_0}$ and Lemma 3.4.7 imply that the sets $\mathrm{Pol}_{\leq N_0}(b_{ij}^{(m)}(\bar{y}_c^h))$ are *dense constructible* subsets of S. But a dense constructible subset of an irreducible noetherian space contains an open dense subset by [GW10, Prop. 10.14]. This shows that the pole order of $b_{ij}^{(m)}(\bar{y}_c^h)$ along x_1 is bounded by N_0, and thus that E is $(X, \overline{X})$-regular singuar. ■

PROOF (OF THEOREM 3.4.4 (C)). Let $(X, \overline{X})$ be a good compactification. By Proposition 3.3.3, we know that a stratified bundle E on X is $(X, \overline{X})$-regular singular if and only if it is divisor-r.s. Assume that E is $(X, \overline{X})$-regular singular, then E is also curve-r.s.: Every closed immersion $i : C \hookrightarrow X$ with C a regular k-curve extends to a morphism $\bar{i} : \overline{C} \to \overline{X}$, so i^*E is $(C, \overline{C})$-regular singular by Proposition 3.2.17. Note that for this argument to work it is essential that $\overline{X} \setminus X$ is a strict normal crossings divisor. It follows that the implication (δ) from Theorem 3.4.4 holds. ■

3.5 The tannakian perspective on regular singularities

We recall the following construction from the theory of tannakian categories; see also Proposition 1.2.6 and Appendix C.

As usual let k be an algebraically closed field of characteristic $p > 0$, and X a smooth, separated, finite type k-scheme. Moreover, let E be a stratified bundle on X, $\omega_0 : \langle E \rangle_{\otimes} \to \mathrm{Vectf}_k$ a neutral fiber functor, and $G := \underline{\mathrm{Aut}}^{\otimes}(\omega_0)$ the associated monodromy group of E. This is a smooth, finite type k-group scheme by Proposition 3.1.8. We write $\rho_{\mathrm{forget}} : \langle E \rangle_{\otimes} \to \mathrm{Coh}(X)$ for the functor which associates to a stratified bundle the underlying coherent sheaf. Then by Theorem C.3.2 it follows that the G_X-torsor $\underline{\mathrm{Isom}}_K^{\otimes}(\omega_0 \otimes_k \mathcal{O}_X, \rho_{\mathrm{forget}})$ is representable by an X-scheme $h_{E,\omega_0} : X_{E,\omega_0} \to X$.

More precisely: The fiber functor ω_0 induces an equivalence of Ind-categories

$$\omega_0' : \mathrm{Ind}(\langle E \rangle_{\otimes}) \to \mathrm{Ind}(\mathrm{Repf}_k\, G) \cong \mathrm{Rep}_k\, G.$$

Note that $\mathrm{Ind}(\mathrm{Repf}_k\, G)$ is equivalent to $\mathrm{Rep}_k\, G$ by Proposition C.2.2, and that the category $\mathrm{Ind}(\langle E \rangle_{\otimes})$ is a subcategory of the category of $\mathcal{O}_X$-quasi-coherent $\mathscr{D}_{X/k}$-modules. We also write

$$\rho_{\mathrm{forget}}' : \mathrm{Ind}(\langle E \rangle_{\otimes}) \to \mathrm{Ind}(\mathrm{Coh}(X)) \cong \mathrm{QCoh}(X)$$

for the extension of ρ_{forget} to Ind-categories. Then Theorem C.3.2 shows that there exists a canonical object A_{ω_0} in $\mathrm{Ind}(\langle E\rangle_{\otimes})$, which is an algebra over the trivial stratified bundle of rank 1, such that $\omega_0'(A_{\omega_0}) = (\mathcal{O}_G, \Delta)$ is the right-regular representation of G, and such that $X_{E,\omega} = \mathbf{Spec}(\rho'_{\mathrm{forget}}(A_{\omega_0}))$. Here $(\mathcal{O}_G, \Delta)$ is $\mathcal{O}_G$ considered as $\mathcal{O}_G$-comodule via the diagonal morphism $\Delta : \mathcal{O}_G \to \mathcal{O}_G \otimes_k \mathcal{O}_G$. Note that by Proposition C.2.6, for any $E' \in \langle E\rangle_{\otimes}$ we have $A_{\omega_0} \otimes_{\mathcal{O}_X} E' \cong A_{\omega_0}^n$ as objects of $\mathrm{Ind}(\langle E\rangle_{\otimes})$, for some $n \geq 0$.

The following proposition shows how regular singularity of E is reflected in the G_X-torsor $h_{E,\omega_0} : X_{E,\omega_0} \to X$:

3.5.1 Proposition. *In addition to the notations from the preceding paragraphs, let $(X, \overline{X})$ be a good partial compactification.*

Then E is regular singular (resp. $(X, \overline{X})$-regular singular) if and only if the $\mathcal{O}_X$-quasi-coherent $\mathscr{D}_{X/k}$-module A_{ω_0} is an increasing union of regular singular stratified bundles (resp. $(X, \overline{X})$-regular singular stratified bundles). □

Proof. Just for this proof "regular singular" either means $(X, \overline{X})$-regular singular or regular singular in the sense of Definition 3.3.1. The argument is the same.

If E is regular singular, then every object of $\langle E\rangle_{\otimes}$ is regular singular by Proposition 3.3.4, and A_{ω_0} is an object of $\mathrm{Ind}(\langle E\rangle_{\otimes})$ by definition. Thus the quasi-coherent $\mathscr{D}_{X/k}$-module A_{ω_0} is an inductive limit of regular singular stratified bundles, and hence also an increasing union of regular singular stratified bundles.

Conversely, if E is a stratified bundle, then $\omega_0'(E)$ is a finite dimensional representation of G, and hence a subrepresentation of $(\mathcal{O}_G, \Delta)^n$ for some n, see Proposition C.2.6. Thus the stratified bundle E is a subobject of $A_{\omega_0}^n$ in $\mathrm{Ind}(\langle E\rangle_{\otimes})$. But by assumption A_{ω_0} is an increasing union of regular singular stratified bundles, so $A_{\omega_0}^n$ is an increasing union of regular singular stratified bundles. This means that E is a substratified bundle of a regular singular stratified bundle and hence regular singular itself according to Proposition 3.3.4. ■

Chapter 4

Finite regular singular stratified bundles

Let k again denote an algebraically closed field of characteristic $p > 0$. In this chapter we study stratified bundles and regular singular stratified bundles having finite monodromy groups. Recall that by dos Santos' theorem (Proposition 3.1.8), a finite monodromy group of a stratified bundle is automatically finite étale, hence constant under our assumption on k. Saavedra [SR72, §VI.1] and dos Santos [dS07] show how stratified bundles with finite monodromy and finite étale coverings are related: If $f : Y \to X$ is a finite étale galois morphism with galois group G, then $f_*\mathcal{O}_Y$ is a stratified bundle on X with monodromy group the constant k-group scheme associated with G, see Proposition 4.1.4. Conversely, if E is a stratified bundle with finite constant monodromy group G, then using techniques going back to Nori, one constructs from E a finite étale galois covering $X_E \to X$ with group G, which trivializes E.

In this chapter we extend these methods and results to regular singular bundles, and show that regular singular bundles with finite monodromy group are related in the above sense to *tamely ramified* étale coverings.

We begin by reviewing some facts about stratified bundles with finite monodromy group.

4.1 Finite stratified bundles

Throughout this section X denotes a smooth, connected, separated, finite type k-scheme.

4.1.1 Definition. An object $E \in \mathrm{Strat}(X)$ is said to be *finite* or is said to have *finite monodromy* if there is a rational point $a \in X(k)$, such that $\pi_1(\langle E\rangle_{\otimes}, \omega_a)$ is finite over k, where ω_a is the neutral fiber functor associated with a. Note that by Proposition 3.1.7 this is equivalent to $\pi_1(\langle E\rangle_{\otimes}, \omega)$ being finite for *all* k-valued fiber functors ω.

Since in our situation $\langle E\rangle_{\otimes}$ always has k-linear fiber functors by Theorem C.1.7, E is finite if and only if every object of $\langle E\rangle_{\otimes}$ is isomorphic to a subquotient of $E^{\oplus n}$ for some n, see Proposition C.2.1. □

4.1.2 Proposition. *If E is a stratified bundle on X, then the following are equivalent:*

(a) *E is finite,*

(b) *$E|_U$ is finite for some open dense $U \subseteq X$,*

(c) *$E|_U$ is finite for any open dense$U \subseteq X$.* □

PROOF. This follows directly from Proposition 3.1.6. ■

4.1.3 Proposition. *If E is a finite stratified bundle on X, then every object of $\langle E\rangle_{\otimes}$ is finite.* □

PROOF. Let $E' \in \langle E\rangle_{\otimes}$. If $\langle E'\rangle_{\otimes} \subseteq \mathrm{Strat}(X)$ is the tannakian subcategory (Definition C.1.5) generated by E', then $\langle E'\rangle_{\otimes}$ naturally is is a tannakian subcategory of $\langle E\rangle_{\otimes}$; more precisely it is the tannakian subcategory of $\langle E\rangle_{\otimes}$ generated by E'. If $\omega : \langle E\rangle_{\otimes} \to \mathrm{Vectf}_k$ is a fiber functor, then Proposition C.2.3 shows that the induced morphism of k-group schemes

$$\pi_1(\langle E\rangle_{\otimes}, \omega) \to \pi_1(\langle E'\rangle_{\otimes}, \omega|_{\langle E'\rangle_{\otimes}})$$

is faithfully flat. Thus, if E is finite, so is E'. ■

The main tool for working with finite stratified bundles is given by the following proposition.

4.1.4 Proposition. *Let $\mathcal{S} \subseteq \mathrm{Strat}(X)$ be a $\otimes$-finitely generated, tannakian subcategory of $\mathrm{Strat}(X)$ (see Definition C.1.6), $\omega : \mathcal{S} \to \mathrm{Vect}_k$ be a fiber functor and $G := \pi_1(\mathcal{S}, \omega)$. Then there exists a smooth G-torsor $h_{\mathcal{S},\omega} : X_{\mathcal{S},\omega} \to X$ with the following properties:*

(a) *Every object of $\mathcal{S}$ is finite if and only if $h_{\mathcal{S},\omega}$ is finite étale.*

From now on assume that G is a finite (thus constant by Proposition 3.1.8) group scheme, and hence $h_{E,\omega}$ finite étale. Then $h_{E,\omega}$ has the following properties.

(b) *An object $E \in \mathrm{Strat}(X)$ is contained in $\mathcal{S}$ if and only if $h^*_{\mathcal{S},\omega}E$ is trivial.*

(c) *If $\mathcal{S}'$ is a second tannakian subcategory of $\mathrm{Strat}(X)$ such that $\mathcal{S}' \subseteq \mathcal{S}$, then there is finite étale morphism g such that the diagram*

$$\begin{tikzcd} X_{\mathcal{S},\omega} \arrow[r, "g"] \arrow[d, "h_{\mathcal{S},\omega}"'] & X_{\mathcal{S}',\omega} \arrow[dl, "h_{\mathcal{S}',\omega}"] \\ X & \end{tikzcd}$$

commutes.

(d) *If ω' is a second fiber functor on $\mathcal{S}$, then there is an X-isomorphism $X_{\mathcal{S},\omega} \xrightarrow{\cong} X_{\mathcal{S},\omega'}$.*

(e) *For $E \in \mathcal{S}$ we have a functorial isomorphism*

$$\omega(E) = H^0(\mathrm{Strat}(Y), h^*_{\mathcal{S},\omega}E) = \left(h^*_{\mathcal{S},\omega}E\right)^{\nabla}.$$

For the general definition of H^0, see Definition C.1.13.

Conversely, if G is a finite constant group scheme and $f : Y \to X$ a G-torsor, i.e. a finite étale galois covering with group G, and $\mathcal{Y}$ the tannakian subcategory of $\mathrm{Strat}(X)$ *generated by those objects which become trivial after pullback allong f, then:*

(f) *$\mathcal{Y} = \langle f_*\mathcal{O}_Y\rangle_{\otimes}$ and there is a fiber functor $\omega_f : \mathcal{Y} \to \mathrm{Vect}_k$, such that $f = h_{\mathcal{Y},\omega_f}$, and such that $\pi_1(\langle f_*\mathcal{O}_Y\rangle_{\otimes}, \omega_f) = G$.*

(g) *If $\mathcal{S} \subseteq \mathrm{Strat}(X)$ is a full tannakian subcategory such that f^*E is trivial for every object $E \in \mathcal{S}$, then $\mathcal{S} \subseteq \mathcal{Y}$, $\pi_1(\mathcal{S}, \omega_f|_{\mathcal{S}})$ is finite constant and there is a morphism $g : Y \to X_{\mathcal{S},\omega}$, such that $f = h_{\mathcal{S},\omega} \circ g$.* □

PROOF. Recall that $\mathcal{S}$ can actually be generated by a single $\otimes$-generator (Definition C.1.6), but we work with $\mathcal{S}$ for notational convenience.

We again use the construction of Section C.3. The main ingredient not intrinsic to the theory of tannakian categories is the fact that if $E \in \mathrm{Strat}(X)$ is a stratified bundle then $\pi_1(\langle E\rangle_{\otimes}, \omega)$ is a smooth k-group scheme by dos Santos' theorem Proposition 3.1.8. In particular, if $\pi_1(\langle E\rangle_{\otimes}, \omega)$ is finite, then it is finite étale and hence constant if k is algebraically closed.

Back to the notations of the proposition: Let $\rho : \mathrm{Strat}(X) \to \mathrm{Coh}(X)$ denote the forgetful functor. To the fiber functor $\omega : \mathcal{S} \to \mathrm{Vectf}_k$ we associate in Section C.3 a quasi-coherent $\mathcal{O}_X$-algebra $A_{\mathcal{S},\omega}$ with $\mathscr{D}_{X/k}$-action, such that $A_{\mathcal{S},\omega}$ corresponds to the right regular representation of G in $\mathrm{Rep}_k G$, and such that

$$\operatorname{Spec} A_{\mathcal{S},\omega} = \underline{\mathrm{Isom}}_k^{\otimes}(\omega, \rho|_{\mathcal{S}}) =: X_{\mathcal{S},\omega}.$$

Write $h_{\mathcal{S},\omega} : X_{\mathcal{S},\omega} \to X$ for this G-torsor. Since G is smooth over k, $h_{\mathcal{S},\omega}$ is smooth, and it is finite if and only if $A_{\mathcal{S},\omega}$ is coherent, if and only if $\mathcal{S} = \langle A_{\mathcal{S},\omega}\rangle_{\otimes}$ if and only if G is finite. If G is finite, then every object of $\mathcal{S}$ is finite by Proposition 4.1.3, because $\mathcal{S}$ admits a $\otimes$-generator. Conversely, if every object of $\mathcal{S}$ is finite, then in particular a $\otimes$-generator of $\mathcal{S}$ is finite, so G is finite. (a) follows.

Now assume that G is a finite étale group scheme on k, hence constant. Then $h_{\mathcal{S},\omega}$ is a finite étale morphism; in particular, $A_{\mathcal{S},\omega}$ is an $\mathcal{O}_X$-algebra in the category $\mathrm{Strat}(X)$. Moreover, the $\mathscr{D}_{X_{\mathcal{S},\omega}/k}$-action on $h^*_{\mathcal{S},\omega}E = E \otimes_{\mathcal{O}_X} A_{\mathcal{S},\omega}$ agrees with the tensor product $\mathscr{D}_{X/k}$-action on $E \otimes_{\mathcal{O}_X} A_{\mathcal{S},\omega}$ via the isomorphism $\mathscr{D}_{X_{E,\omega}/k} \xrightarrow{\cong} h^*_{\mathcal{S},\omega}\mathscr{D}_{X/k}$. In other words, the pull-back functor $h^*_{E,\omega}$ factors through

$$\mathrm{Strat}(X) \to \mathrm{Strat}(X), E \mapsto E \otimes A_{\mathcal{S},\omega}.$$

Now everything follows fairly directly from general theory: (b) and (c) follow from Proposition C.2.7; (d) follows from the fact that $\underline{\mathrm{Isom}}_k^{\otimes}(\omega, \omega')$ is an fpqc-torsor on k, but k is algebraically closed, so the torsor is trivial.

Statement (e), follows from Proposition C.3.5.

For (f), note that $f_*\mathcal{O}_Y$ is a stratified bundle on X; more precisely, it is an $\mathcal{O}_X$-algebra in $\mathrm{Strat}(X)$, see e.g. Example D.2.4, or Proposition 3.1.2. As above we see that the $\mathscr{D}_{Y/k}$-structure on f^*E agrees with the $\mathscr{D}_{X/k}$-structure on $E \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y$ via the isomorphism $\mathscr{D}_{Y/k} \xrightarrow{\cong} f^*\mathscr{D}_{X/k}$. Since f is galois, $f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \cong f_*\mathcal{O}_Y^{\deg f}$, so $f_*\mathcal{O}_Y$ is trivialized on Y. Conversely, if $f^*E = E \otimes f_*\mathcal{O}_Y$ is trivial, i.e. $E \otimes f_*\mathcal{O}_Y \cong f_*\mathcal{O}_Y^n$ for some n, then E is a $\mathscr{D}_{X/k}$-submodule of $f_*\mathcal{O}_Y^n$, so we have proven that the subcategory of $\mathrm{Strat}(X)$ spanned by bundles E trivialized on Y is precisely $\langle f_*\mathcal{O}_Y \rangle_\otimes$. We define the functor $\omega_f : \langle f_*\mathcal{O}_Y \rangle_\otimes \to \mathrm{Vectf}_k$ by $\omega_f(E) = H^0(\mathrm{Strat}(Y), f^*E) = (E \otimes f_*\mathcal{O}_Y)^\nabla$ (see Definition C.1.13 for the general definition of H^0), and this functor is faithful and exact since f^*E is trivial for all E. Finally, the fundamental group $\pi_1(\langle f_*\mathcal{O}_Y \rangle_\otimes, \omega)$ is the constant group scheme associated with G: Since $Y \times_X Y \cong Y \times_k G$, we see that $\omega_f(f_*\mathcal{O}_Y)$ is the right regular representation of G.

Lastly, (g) follows immediately from Proposition C.2.7. ∎

4.1.5 Corollary. *If $f : Y \to X$ is a finite étale morphism, and $f' : Y' \to X$ its galois closure with galois group G, then*

$$\mathcal{Y} := \langle f_*\mathcal{O}_Y \rangle_\otimes = \langle f'_*\mathcal{O}_{Y'} \rangle_\otimes \subseteq \mathrm{Strat}(X),$$

$f' = h_{\mathcal{Y},\omega_{f'}}$, and $f'_*\mathcal{O}_{Y'}$ is finite with monodromy group the constant k-group scheme associated with G. □

PROOF. Clearly $\mathcal{Y} \subseteq \langle f'_*\mathcal{O}_{Y'} \rangle_\otimes$ by Proposition 4.1.4, (b), so if $h_{\mathcal{Y},\omega_{f'}} : X_{\mathcal{Y},\omega_{f'}} \to X$ is the associated galois étale morphism, then $h_{\mathcal{Y},\omega_{f'}}$ factors through f (because $h^*_{\mathcal{Y},\omega_{f'}}(f_*\mathcal{O}_Y)$ is trivial), and then by Proposition 4.1.4, (g), there is a morphism $Y' \to X_{\mathcal{Y},\omega_{f'}}$, such that the diagram

$$\begin{tikzcd}
X_{\mathcal{Y},\omega_{f'}} \arrow[dr] \arrow[ddr, "h_{\mathcal{Y},\omega_{f'}}"'] & & Y' \arrow[ll] \arrow[dl] \arrow[ddl, "f'"] \\
& Y \arrow[d, "f"] & \\
& X &
\end{tikzcd}$$

commutes. But since f' is the galois closure of f, it follows that $Y' \to X_{\mathcal{Y},\omega_{f'}}$ is an isomorphism. This shows that $\pi_1(\mathcal{Y}, \omega'_f) = G$. ∎

4.1.6 Definition. Let $E \in \mathrm{Strat}(X)$ be a stratified bundle and $\omega : \langle E \rangle_\otimes \to \mathrm{Vectf}_k$ a fiber functor. We write $h_{E,\omega} : X_{E,\omega} \to X$ for the smooth $\pi_1(\langle E \rangle_\otimes, \omega)$-torsor associated with $\langle E \rangle_\otimes$ and ω in Proposition 4.1.4. If E is finite, then we call the finite galois covering $h_{E,\omega}$ the *universal trivializing torsor associated with E and ω.* □

4.1.7 Remark. A caution is in order: If E is not a finite stratified bundle, then $h^*_{E,\omega}E$ is *not necessarily* trivial in $\mathrm{Strat}(X_{E,\omega})$; it is true that the $\mathscr{D}_{X/k}$-module $E \otimes A_{E,\omega}$ is isomorphic to $A_{E,\omega}^{\mathrm{rank}\, E}$, but $h^*_{E,\omega}E$ also carries an action of the relative differential operators, which were trivial in the étale case. The fact that there are no "new" differential operators acting on $h^*_{E,\omega}E$ when E is finite, makes the trivializing torsors useful in our context. □

4.1.8 Proposition ([EL11, Lemma 1.1]). *A stratified bundle E on X is finite if and only if there exists a finite étale covering $f : Y \to X$, such that $f^*E \in \mathrm{Strat}(Y)$ is trivial.* □

Proof. If E is finite, then it is trivialized by the universal trivializing torsor associated with E and a fiber functor ω.

Conversely, if $f : Y \to X$ is finite étale such that f^*E is trivial. Then $E \subseteq f_*f^*E \subseteq f_*\mathcal{O}_Y^n$ as stratified bundles for some n. Without loss of generality we may assume that f is galois étale with finite group G, so by Proposition 4.1.4 we know that $f_*\mathcal{O}_Y$ is a finite stratified bundle with monodromy the constant k-group scheme associated with G. Then by Proposition 4.1.3 we know that $f_*\mathcal{O}_Y^n$, and finally $E \subseteq f_*\mathcal{O}_Y^n$ are finite. ■

4.1.9 Corollary. *If E is a stratified bundle on X, then the following are equivalent:*

(a) *E is finite,*

(b) *E is étale locally finite, i.e. for any étale morphism $f : U \to X$, with U of finite type over k, the stratified bundle f^*E is finite,*

(c) *there exists some étale morphism $f : U \to X$, with U of finite type over k, such that f^*E is finite.* □

Proof. The main ingredient is that if $f : U \to X$ is any étale morphism with U of finite type over k, then there is a nonempty open $V \subseteq X$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite étale, since X is reduced.

Thus, if $f : U \to X$ is étale with U of finite type over k, let $V \subseteq X$ be as above. Then E is finite if and only if $E|_V$ is finite by Proposition 4.1.2, and by Proposition 4.1.8, $E|_V$ is finite if and only if $(f|_{f^{-1}(V)})^*E|_V = (f^*E)|_{f^{-1}(V)}$ is finite. But again by Proposition 4.1.2, f^*E is finite if and only if $(f^*E)|_{f^{-1}(V)}$ is finite, so we are done. ■

4.2 Exponents of finite $(X, \overline{X})$-regular singular stratified bundles

We continue to denote by k an algebraically closed field of characteristic $p > 0$.

4.2.1 Proposition. *Let $(X, \overline{X})$ be a good partial compactification, and E a stratified bundle on X.*

(a) *E is $(X, \overline{X})$-regular if and only if E is $(X, \overline{X})$-regular singular and*

$$\mathrm{Exp}_{(X,\overline{X})}(E) = \{0\} \subseteq \mathbb{Z}_p/\mathbb{Z}.$$

(b) *If E is $(X, \overline{X})$-regular singular and finite, then every element of*

$$\mathrm{Exp}_{(X,\overline{X})}(E) \subseteq \mathbb{Z}_p/\mathbb{Z}$$

is torsion. □

4.2.2 Remark. (a) Proposition 4.2.1, (a) states a major difference to the situation in characteristic 0: If $\mathrm{char}(k) = 0$, then there exist regular singular flat connections on X with exponents 0 which are not regular, since the residues can be nilpotent but nontrivial. As the proof of the proposition shows, analogous objects do not exist in our characteristic $p > 0$ context, essentially because of the existence of the decomposition Proposition 2.7.1.

(b) To illustrate the difference between characteristic 0 and characteristic p mentioned in (a), let $\overline{X} = \operatorname{Spec} k[t]$, $X = \operatorname{Spec} k[t^{\pm 1}]$ and assume that the $\mathcal{O}_{\overline{X}}$-module $\overline{E} := \mathcal{O}_{\overline{X}} \oplus \mathcal{O}_{\overline{X}}$ with basis e_1, e_2 is given an action of $\mathscr{D}_{\overline{X}/k}(\log 0)$ with $\delta_t^{(p^n)}(e_1) = 0$ and $\delta_t^{(p^n)}(e_2) = f_n e_1$ with $f_n \in k[t]$. We claim that $\delta_t^{(p^n)}(\overline{E}) \subseteq t^{p^n}\overline{E}$ for every n. Indeed, since in $\mathscr{D}_{\overline{X}/k}(\log 0)$ we have $(\delta_t^{(p^n)})^p = \delta_t^{(p^n)}$, it follows that

$$(\delta_t^{(p^n)})^{p-1}(f_n) = f_n$$

and hence $f_n \in k[t^{\pm p^n}]$. Thus: every $\delta_t^{(p^n)}$ acts trivially on $\overline{E}|_0$, and the $\mathscr{D}_{\overline{X}/k}(\log D)$-action extends to a $\mathscr{D}_{\overline{X}/k}$-action on $\overline{E}$.

(c) The argument for Proposition 4.2.1, (b) given below actually shows more: If there exists, perhaps after replacing $(X, \overline{X})$ by an equivalent partial compactification (see Appendix A), a good partial compactification $(Y, \overline{Y})$ and a finite morphism $\bar{f} : \overline{Y} \to \overline{X}$ such that $f := \bar{f}|_Y$ is a finite étale morphism $Y \to X$, and such that f^*E is $(Y, \overline{Y})$-regular, then the exponents of E are torsion in $\mathbb{Z}_p/\mathbb{Z}$.

(d) Tempting as it might seem, a converse to Proposition 4.2.1, (b) is *not* true: If $\overline{E}$ is a stratified bundle on $\overline{X}$ which is not finite, then $\overline{E}|_X$ is a $(X, \overline{X})$-regular singular stratified bundle with torsion exponents, but clearly $\overline{E}|_X$ is not finite.

What is true however is the following: After perhaps replacing $(X, \overline{X})$ by an equivalent good partial compactification (see Appendix A), there exists a good partial compactification $(Y, \overline{Y})$, a finite morphism $\bar{f} : \overline{Y} \to \overline{X}$, tamely ramified over $\overline{X} \setminus X$, such that $f := \bar{f}|_Y$ is a finite étale morphism $Y \to X$, and such that f^*E is $(Y, \overline{Y})$-regular. This follows from the argument in the proof of Proposition 4.3.2. □

PROOF. Write $D := \overline{X} \setminus X$.

(a) We may take care of each component of D separately, so without loss of generality we may assume that D is irreducible. If E is $(X, \overline{X})$-regular, then, by definition, E can be extended to an $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}$-module $\overline{E}$, which is then locally free of finite rank. In particular E is regular singular, and clearly the exponents of $\overline{E}$ are 0.

Conversely, assume that E is an $(X, \overline{X})$-regular singular stratified bundle with $\mathrm{Exp}_{(X,\overline{X})}(E) = 0$. By Theorem 3.2.9 we may assume that there exists a locally free $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module extending E such that the exponents of $\overline{E}$ along D are 0. Then $\overline{E}|_D = F_0$ (see Proposition 2.7.1), which in local coordinates $x_1, \dots, x_n$, such that $D = V(x_1)$, means that around D, $\delta_{x_1}^{(m)}(\overline{E}) \subseteq x_1\overline{E}$ for all m. We prove that in fact $\delta_{x_1}^{(p^m)}(\overline{E}) \subseteq x_1^{p^m}\overline{E}$ for all $m \geq 0$. This would prove that the $\mathscr{D}_{\overline{X}/k}(\log D)$-action extends to a $\mathscr{D}_{\overline{X}/k}$-action.

Since we know that $\partial_{x_1}^{(1)}(\overline{E}) \subseteq x_1\overline{E}$, that the logarithmic connection given by the action of the operators of order ≤ 1 comes from an actual non-logarithmic connection ∇_1 on $\overline{E}$. Write $\overline{E}_1 = \overline{E}^{\nabla_1}$ for the subsheaf of $\overline{E}$ given by sections e such that $\partial_{x_1}^{(1)}(e) = 0$. Then $\overline{E}_1$ is an $\mathcal{O}_{\overline{X}^{(1)}}$-module, and since ∇_1 is a connection, Cartier's theorem [Kat70, Thm. 5.1] shows that $F_{\overline{X}/k}^*\overline{E}_1 \cong \overline{E}$, and that ∇_1 is the canonical connection on $F_{\overline{X}/k}^*\overline{E}_1$. In particular, $\overline{E}_1$ is locally free over $\mathcal{O}_{\overline{X}^{(1)}}$ with $\operatorname{rank}\overline{E}_1 = \operatorname{rank}\overline{E}$. Moreover, $\overline{E}_1$ carries a $\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$-action induced by the one on $\overline{E}$: $\delta_{x_1}^{(p^m)}$ acts as an operator of order p^{m-1} for all m, and the exponents are still 0, since pulling back along $F_{\overline{X}/k}^*$ multiplies exponents by p, see Corollary 3.2.21 and its proof. More precisely, if $y_1, \dots, y_n$ are the coordinates of $\overline{X}^{(1)}$ corresponding to $x_1, \dots, x_n$, then $\delta_{x_1}^{(p^m)}$ acts as $\delta_{y_1}^{(p^{m-1})}$. We can apply Proposition 2.7.1 to the $\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$-module $\overline{E}_1$ to see that $\delta_{y_1}^{(1)}(\overline{E}_1) \subseteq y_1\overline{E}_1$. Under the isomorphism $F_{\overline{X}/k}^*\overline{E}_1 \cong \overline{E}$, this means that $\delta_{x_1}^{(p)}(\overline{E}) \subseteq x_1^p\overline{E}$, so we can give meaning to the operation of $\partial_{x_1}^{(p)}$ on $\overline{E}$. Denoting by $\overline{E}_2$ the subsheaf of $\overline{E}$ consisting of sections annihilated by $\partial_{x_1}^{(1)}$ and $\partial_{x_1}^{(p)}$, we can redo the same argument to see that $\delta_{x_1}^{(p^2)}(\overline{E}) \subseteq x_1^{p^2}\overline{E}$ and so on.

(b) Assume that E is finite and $(X, \overline{X})$-regular singular, and let $\overline{E}$ be a locally free $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module extending E. To show that $\operatorname{Exp}_{(X,\overline{X})}(\overline{E}) \subseteq \mathbb{Z}_p \cap \mathbb{Q}$, we may assume that D is irreducible. Let $f : Y \to X$ be a finite étale covering such that f^*E is trivial. Then, after shrinking X further, we may assume that there is a normal scheme $\overline{Y}$ such that $Y \subseteq \overline{Y}$, and a finite map $\bar{f} : \overline{Y} \to \overline{X}$, restricting to f over X. Shrinking $\overline{X}$ around the generic point η of D, we may further assume that $\overline{Y}$ is smooth and that $\bar{f}^{-1}(D)$ is the disjoint union of smooth divisors $D'_1, \dots, D'_n$. Then f^*E is $(Y, \overline{Y})$- regular singular, since a $\mathscr{D}_{\overline{Y}/k}(\log \bar{f}^{-1}(D))$-extension of f^*E is given by $\overline{f}^*\overline{E}$. We know that $\bar{f}^*\overline{E}$ has exponents in $\mathbb{Z}$, so it suffices to show that the exponent of $\bar{f}^*\overline{E}$ along D'_i are the exponents of $\overline{E}$ along D multiplied by the ramification index of D'_i over D. This is now a question about finite extensions $g : R \hookrightarrow S$ of discrete valuation rings, with separable extension of the fraction fields: Since the extension of fraction fields is separable, g induces an isomorphism $\mathscr{D}_{S/k} \otimes \operatorname{Frac} S \to \mathscr{D}_{R/k} \otimes_R \operatorname{Frac} S$. If x is a uniformizer for R, y a uniformizer for S, and $uy^e = x$, with $u \in S^\times$, then I claim that

$$\delta_y^{(p^m)} = \sum_{c+d=m} \binom{e}{p^c} \delta_x^{(p^d)} \mod y \cdot \mathscr{D}_{R/k}(\log D) \otimes_R S. \tag{4.1}$$

This would complete the proof because if $a \in E \otimes R$ is an element such that $\delta_x^{(p^d)}$ acts via

$$\delta_x^{(p^d)}(a) = \binom{\alpha}{p^d} a \mod xE$$

then if (4.1) holds, Lemma 1.1.9, (c) implies that

$$\delta_y^{(p^m)}(a) = \sum_{c+d=m} \binom{e}{p^c}\binom{\alpha}{p^d} a + yE \otimes S = \binom{e\alpha}{p^m} a + yE.$$

To show that (4.1) holds, we compute

$$\begin{aligned}
\delta_y^{(p^m)}(x^r) &= \delta_y^{(p^m)}(u^r y^{er}) \\
&= y^{p^m} \sum_{\substack{a+b=p^m \\ a,b\geq 0}} \partial_y^{(a)}(u^r)\partial_y^{(b)}(y^{er}) \\
&= \binom{er}{p^m} u^r y^{er} + \sum_{\substack{a+b=p^m \\ a>0,b\geq 0}} \delta_y^{(a)}(u^r)\binom{er}{b} y^{er} \\
&= \binom{er}{p^m} x^r + \sum_{\substack{a+b=p^m \\ a>0,b\geq 0}} \binom{er}{b} x^r \frac{\delta_y^{(a)}(u^r)}{u^r} \qquad (4.2)
\end{aligned}$$

and

$$\sum_{c+d=m} \binom{e}{p^c} \delta_x^{(p^d)}(x^r) = \binom{er}{p^m} x^r,$$

where we have again used Lemma 1.1.9, (c).

Finally note that the difference $\theta := \delta_y^{(p^m)} - \sum_{c+d=m} \binom{e}{p^c}\delta_x^{(p^d)}$ is a differential operator in $\mathscr{D}_{R/k}(\log D) \otimes_R S$, which is divisible by y: In fact we computed $\theta(x^r)$ in (4.2), and

$$\frac{\delta_y^{(a)}(u^r)}{u^r}$$

is always contained in (y), so for all r, we see that $\theta(x^r) \in (x^r y)$, which proves that $\theta \in y\mathscr{D}_{R/k}(\log D) \otimes_R S$. ∎

4.3 Tamely ramified coverings as regular singular stratified bundles

We continue to denote by k an algebraically closed field of characteristic $p > 0$. Let X be a smooth, separated, finite type k-scheme. The goal of this section is to relate the structure of an étale covering $f : Y \to X$ with the structure of the finite stratified bundle $f_*\mathscr{O}_Y$ on X, see Corollary 4.1.5. We beginn with the following easy lemma.

4.3.1 Lemma. *Let $f : Y \to X$ be a finite étale covering.*

(a) *f is the trivial covering if and only if the stratified bundle $f_*\mathscr{O}_Y$ is trivial.*

(b) *If $g : Z \to X$ is a second étale covering such that the projection $Z \times_X Y \to Z$ is the trivial covering, then g factors (nonuniquely) through f, i.e. there is a commutative diagram*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[dr, "g"'] & Y \arrow[d, "f"] \\ & X \end{tikzcd}$$

□

PROOF. In Corollary 4.1.5 we saw that the galois closure $f' : Y' \to Y$ of f is a universal trivializing torsor associated via tannaka theory with the tannakian category $\langle f_* \mathcal{O}_Y \rangle_\otimes$ and some fiber functor, so (a) follows. (Of course one can also give a direct proof: The point is that if $f_* \mathcal{O}_Y \cong \mathcal{O}_X^n$ is a morphism of $\mathscr{D}_{X/k}$-modules, then it automatically is an isomorphism of $\mathcal{O}_X$-algebras).

Part (b) is a consequence of Proposition 4.1.4, but it is easier to note that under our assumption the trivial covering $Z \times_X Y \to Z$ has a section σ

$$\begin{tikzcd} \coprod Z \arrow[r, "g_Y"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ Z \arrow[u, bend left, "\sigma"] \arrow[ur, bend left, "h"] \arrow[r, "g"] & X, \end{tikzcd}$$

so we can define $h := (g_Y) \circ \sigma$. ■

4.3.2 Proposition. *Let $(X, \overline{X})$ be a good partial compactification and $f : Y \to X$ a finite étale morphism. The following are equivalent:*

(a) *f is tamely ramified with respect to the strict normal crossings divisor $\overline{X} \setminus X$.*

(b) *$f_* \mathcal{O}_Y$ is $(X, \overline{X})$-regular singular.* □

PROOF. (a) implies (b) by Corollary 3.2.18.

For the converse, we may assume that Y is galois étale: If the galois closure of Y is tamely ramified with respect to $D := \overline{X} \setminus X$, then so is Y, see Proposition B.1.10.

Let $\eta_1, \dots, \eta_n$ be the generic points of D, and $\overline{X}_1, \dots, \overline{X}_n$ open neighborhoods of $\eta_1, \dots, \eta_n$, such that $\eta_j \in \overline{X}_i$ if and only if $j = i$. Define $X_i := \overline{X}_i \cap X$ and $Y_i := f^{-1}(X_i)$. To show that $f : Y \to X$ is tamely ramified with respect to $(X, \overline{X})$, it suffices to show that $f_i := f|_{Y_i} : Y_i \to X_i$ is tamely ramified with respect to η_i for $i = 1, \dots, n$. Hence we may assume that D is irreducible with generic point η, and we may always replace $\overline{X}$ by an open neighborhood $\overline{X}'$ of η, X by $\overline{X}' \cap X$ and Y by the preimage of $X \cap \overline{X}'$.

To show that $f : Y \to X$ is tamely ramified with respect to η, it is enough to show that (perhaps after shrinking $\overline{X}$ around η) there exists a finite étale morphism $g : W \to X$, tamely ramified with respect to η, such that the pullback $W \times_X Y \to W$ is the trivial covering, because then f is dominated by a covering tamely ramified with respect to η and hence tame over η itself.

For readability we write $E := f_*\mathcal{O}_Y$. By Proposition 4.2.1, the exponents of E are torsion in $\mathbb{Z}_p/\mathbb{Z}$, which means that if $\overline{E}$ is a locally free $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module extending E, then the exponents of $\overline{E}$ are in $\mathbb{Q}\cap\mathbb{Z}_p$. This means we can find a *tamely ramified* finite étale covering $\bar{g} : \operatorname{Spec} R \to \operatorname{Spec} \mathcal{O}_{\overline{X},\eta}$ of the discrete valuation ring $\mathcal{O}_{\overline{X},\eta}$, such that the exponents of $\overline{E}\otimes_{\mathcal{O}_{\overline{X},\eta}} R$ along the components of $\bar{g}^{-1}(\eta)$ are in $\mathbb{Z}$.

Indeed, let x_1 be a uniformizer for $\mathcal{O}_{\overline{X},\eta}$. If $\alpha \in (\mathbb{Z}_p\cap\mathbb{Q})\setminus\mathbb{Z}$ has the property that $F_\alpha \neq 0$ in the decomposition of $\overline{E}|_D$ from Proposition 2.7.1, say $\alpha = r/s$, with $(s,p) = 1$ and $(r,s) = 1$, then the Kummer covering given by $\bar{g} : \mathcal{O}_{\overline{X},\eta} \to R := \mathcal{O}_{\overline{X},\eta}[t]/(t^s - x_1)$ is tamely ramified, and the exponents of $\bar{g}^*\overline{E}_\eta$ along $\bar{g}^{-1}(\eta)$ will be the exponents of $\overline{E}$ along η multiplied by s. We saw this in the proof of Proposition 4.2.1. Repeating this process for all noninteger exponents, we obtain the desired tame covering, which we will also denote by $\bar{g} : \operatorname{Spec} R \to \operatorname{Spec}\mathcal{O}_{\overline{X},\eta}$, such that the exponents of $\bar{g}^*\overline{E}$ are contained in $\mathbb{Z}\subseteq\mathbb{Z}_p$.

Write g for the restriction of $\bar{g}$ to $\bar{g}^{-1}(\operatorname{Spec} k(X))$. By construction g is étale, and by Lemma B.1.9 we may also assume that g is galois étale. To summarize the notation:

$$
\begin{tikzcd}
\operatorname{Spec} R \arrow[r, "\bar{g}", "\text{tame}"'] & \operatorname{Spec}\mathcal{O}_{\overline{X},\eta} \\
\operatorname{Spec}(\operatorname{Frac} R) \arrow[u, hook, "\text{open}"] \arrow[r, "g", "\text{galois étale}"'] & \operatorname{Spec} k(X) \arrow[u, hook, "\text{open}"']
\end{tikzcd}
$$

The fact that $\bar{g}^*\overline{E}_\eta$ has integer exponents then implies (by Proposition 4.2.1) that we find a stratified bundle $\overline{E}'$ on R, such that $\overline{E}'|_{\operatorname{Spec}(\operatorname{Frac} R)} \cong g^*E|_{\operatorname{Spec} k(X)}$ as stratified bundles on $k(X) = \operatorname{Frac}\mathcal{O}_{\overline{X},\eta}$.

Since g is finite, we may actually spread it out to an open neighborhood $\overline{U}$ of $\eta\in\overline{X}$, i.e. after replacing $\overline{X}$ by $\overline{U}$, X by $\overline{U}\cap X$, Y by $f^{-1}(\overline{U}\cap X)$ we find a finite morphism $\bar{g} : \overline{V}\to\overline{X}$, such that writing $V := \bar{g}^{-1}(X)$, and $g := \bar{g}|_V$, it follows that g is étale, $\bar{g}$ tamely ramified over η, and such that there is a stratified bundle $\overline{E}'$ on $\overline{V}$ restricting to the finite bundle g^*E. Note that Proposition 3.1.6 implies that $\overline{E}'$ is also finite. But this means we find a finite *étale* morphism $\bar{h} : \overline{W}\to\overline{V}$, such that $h^*\overline{E}'$ is trivial. Writing $W := \bar{h}^{-1}(V)$, we see that h^*g^*E is trivial and gh is tame. Finally, since $E = f_*\mathcal{O}_Y$, this means by Lemma 4.3.1 that the covering

$$Y\times_X W\to W$$

obtained by pulling back $f : Y\to X$ along $gh : W\to X$ is trivial, which is what we wanted to prove. ∎

Using Proposition 4.3.2 we can derive a similar statment for the general notion of regular singular stratified bundles from Definition 3.3.1:

4.3.3 Proposition. *Let X be a smooth, separated, finite type k-scheme and $f : Y\to X$ a finite étale morphism. The following are equivalent:*

(a) *f is tamely ramified (in the sense of Definition B.2.1).*

(b) *f_*E is regular singular for every $E \in \mathrm{Strat}^{\mathrm{rs}}(X)$.*

(c) *$f_*\mathcal{O}_Y$ is regular singular.* □

PROOF. If (a) holds, then (b) follows by Proposition 3.3.6. (b) trivially implies (c), so assume that (c) holds. We need to show that $f : Y \to X$ is tamely ramified. By Definition B.2.1 this is equivalent to f being tamely ramified with respect to every good partial compactification $(X, \overline{X})$ of X. But if $f_*\mathcal{O}_Y$ is regular singular, then it is $(X, \overline{X})$-regular singular for every good partial compactification $(X, \overline{X})$, and (a) holds according to Proposition 4.3.2. ■

4.4 Finite $(X, \overline{X})$-regular singular stratified bundles

It is now easy to derive the main results about finite regular singular stratified bundles relative to a fixed good partial compactification:

4.4.1 Theorem. *Let $(X, \overline{X})$ be a good partial compactification and E a stratified bundle on X. Then the following are equivalent:*

(a) *E is finite and $(X, \overline{X})$-regular singular.*

(b) *There is a finite* galois *étale morphism $f : Y \to X$, tamely ramified with respect to $\overline{X} \setminus X$, such that $f^*E \in \mathrm{Strat}(Y)$ is trivial.*

(c) *There is a finite étale morphism $f : Y \to X$, tamely ramified with respect to $\overline{X} \setminus X$, such that $f^*E \in \mathrm{Strat}(Y)$ is trivial.* □

PROOF. Let $\omega : \langle E \rangle_{\otimes} \to \mathrm{Vectf}_k$ be a fiber functor and $G := \pi_1(\langle E \rangle_{\otimes}, \omega)$. Assume (a). Then as explained before, e.g. in Proposition 4.1.8, the G-torsor $h_{E,\omega} : X_{E,\omega} \to X$ is finite galois étale, and it trivializes E. Proposition 3.5.1 then shows that $(h_{E,\omega})_*\mathcal{O}_{X_{E,\omega}}$ is $(X, \overline{X})$-regular singular, and Proposition 4.3.2 shows that $h_{E,\omega}$ is a tamely ramified galois covering with with respect to $\overline{X} \setminus X$.

Clearly (b) implies (c), so it only remains to prove that (c) implies (a): Assume that $f : Y \to X$ is tamely ramified with respect to $\overline{X} \setminus X$ and assume that f trivializes E. Then $f_*\mathcal{O}_Y$ is $(X, \overline{X})$-regular singular according to Proposition 4.3.2, and E is a substratified bundle of the $(X, \overline{X})$-regular singular stratified bundle $f_*f^*E = f_*\mathcal{O}_Y^{\mathrm{rank}\, E}$, which shows that E itself is $(X, \overline{X})$-regular singular. ■

The proof of the theorem actually shows more:

4.4.2 Corollary. *With the notation from Theorem 4.4.1, the following are equivalent:*

(a) *E is finite and $(X, \overline{X})$-regular singular,*

(b) *If ω is any neutral fiber functor for $\langle E \rangle_{\otimes}$, then the trivializing torsor $h_{E,\omega} : X_{E,\omega} \to X$ is finite étale and tamely ramified with respect to $\overline{X} \setminus X$.*

Moreover, if one of the above holds, then if $f : Y \to X$ is any finite galois covering, such that f^*E is trivial, and $h : Y' \to X$ the maximal subcovering which is tame with respect to $\overline{X} \setminus X$, then h^*E is trivial. □

PROOF. The first statement follows directly from the proof of the theorem. For the second statement, note that f factors through $h_{E,\omega} : X_{E,\omega} \to X$: Since f^*E is trivial, $f^*(h_{E,\omega})_*\mathcal{O}_{X_{E,\omega}}$ is also trivial. By Lemma 4.3.1 this means that the pullback covering $Y \times_X X_{E,\omega} \to Y$ is the trivial covering, and that f indeed factors through $h_{E,\omega}$. But $h_{E,\omega}$ is tamely ramified with respect to $\overline{X} \setminus X$, so $h' : Y' \to X$ also factors through $h_{E,\omega}$, which proves that h'^*E is trivial.

We also obtain a generalization of [dS07, Prop. 13]. For a k-group scheme G, denote by G^0 the connected component of the origin, and write $\pi_0(G) := G/G^0$. This is an étale group scheme over k.

4.4.3 Corollary. *Let $(X, \overline{X})$ be a good partial compactification, and $D := \overline{X} \setminus X$. Let $x \in X$ be a closed point. Write $\Pi^{\mathrm{rs}}_{(X,\overline{X})} := \pi_1(\mathrm{Strat}^{\mathrm{rs}}((X, \overline{X})), \omega_x)$, for the fiber functor ω_x associated with x. Then there is a quotient map of k-group schemes*

$$\phi : \Pi^{\mathrm{rs}}_{(X,\overline{X})} \twoheadrightarrow \pi_1^D(X, x),$$

where $\pi_1^D(X, x)$ is the constant group scheme associated with the profinite group classifying étale coverings of X which are tame with respect to D. This map factors uniquely through the canonical quotient $\Pi^{\mathrm{rs}}_{(X,\overline{X})} \twoheadrightarrow \pi_0(\Pi^{\mathrm{rs}}_{(X,\overline{X})})$, and induces a continuous isomorphism

$$\pi_0(\Pi^{\mathrm{rs}}_{(X,\overline{X})}) \xrightarrow{\sim} \pi_1^D(X, x).$$

□

PROOF. To simplify notation, lets write $\Pi_0 := \pi_0(\Pi^{\mathrm{rs}}_{(X,\overline{X})})$. We first make some general remarks about the category $\mathrm{Repf}_k(\Pi_0)$. The quotient $\Pi^{\mathrm{rs}}_X \twoheadrightarrow \Pi_0$ induces an embedding of tannakian categories $\mathrm{Repf}_k(\Pi_0) \hookrightarrow \mathrm{Repf}_k(\Pi^{\mathrm{rs}}_{(X,\overline{X})})$ such that $\Pi_0 = \mathrm{Aut}^{\otimes}(\omega_x|_{\mathrm{Repf}_k(\Pi_0)})$. The group scheme Π_0 is the inverse limit of the finite type k-group schemes $\pi_1(\langle E\rangle_{\otimes}, \omega_x)$ for $E \in \mathrm{Repf}_k(\Pi_0)$. Note that by dos Santos' theorem Proposition 3.1.8 the groups $\pi_1(\langle E\rangle_{\otimes}, \omega_x)$ are étale over k, and thus also finite étale. Hence Π_0 is pro-étale, and since k is algebraically closed even profinite. On the other hand, if E is an object of $\mathrm{Repf}_k(\Pi^{\mathrm{rs}}_{(X,\overline{X})})$, such that $\pi_1(\langle E\rangle_{\otimes}, \omega_x)$ is finite, then the quotient $\Pi^{\mathrm{rs}}_{(X,\overline{X})} \twoheadrightarrow \pi_1(\langle E\rangle_{\otimes}, \omega_x)$ factors through Π_0, by definition of the functor π_0.

In other words, the essential image of the inclusion functor $\mathrm{Repf}_k(\Pi_0) \hookrightarrow \mathrm{Repf}_k(\Pi^{\mathrm{rs}}_{(X,\overline{X})})$ is the smallest tannakian subcategory of $\mathrm{Repf}_k(\Pi^{\mathrm{rs}}_{(X,\overline{X})})$ containing every object E with $\pi_1(\langle E\rangle_{\otimes}, \omega_x)$ finite. In fact, it is the union of $\langle E\rangle_{\otimes}$ where E runs through the finite objects.

Let $\pi_1^D(X, x) \twoheadrightarrow G$ be a finite quotient, corresponding to a finite galois covering $f : Y \to X$ tamely ramified with respect to D. We have seen in Proposition 4.1.4 and Theorem 4.4.1 that f gives rise to an $(X, \overline{X})$-regular singular bundle $f_*\mathcal{O}_Y$ with $\pi_1(\langle f_*\mathcal{O}_Y\rangle_{\otimes}, \omega_f) = G$ (abusing notation, we write G also for the constant k-group scheme associated to G. Recall that k is always algebraically closed). Here $\omega_f : \langle f_*\mathcal{O}_Y\rangle_{\otimes} \to \mathrm{Vectf}_k$ is the fiber functor given by

$E \mapsto (f^*E)^{\nabla}$. If we consider $\langle f_*\mathcal{O}_Y\rangle_{\otimes}$ as a tannakian subcategory of $\mathrm{Repf}_k \Pi_0$, then $\omega_f = \omega_x|_{\langle f_*\mathcal{O}_Y\rangle_{\otimes}}$.

This means that we get a surjective morphism $\Pi_0 \twoheadrightarrow G$ for every finite quotient G of $\pi_1^D(X,x)$, i.e. a surjective morphism $\Pi_0 \twoheadrightarrow \pi_1^D(X,x)$, and consequently inclusions of categories

$$\mathrm{Repf}_k\left(\pi_1^D(X,x)\right) \hookrightarrow \mathrm{Repf}_k(\Pi_0) \hookrightarrow \mathrm{Repf}_k(\Pi^{\mathrm{rs}}_{(X,\overline{X})}),$$

commuting with the forgetful functors.

Finally, if E is an object of $\mathrm{Repf}_k(\Pi_0)$, then it has finite mondodromy, and $\langle E\rangle_{\otimes} = \langle f_*\mathcal{O}_Y\rangle_{\otimes}$ for some finite étale covering $f : Y \to X$, tame with respect to D (in fact f can be taken to be the trivializing torsor associated to E and ω_x). But this means that $E \in \mathrm{Repf}_k\left(\pi_1^D(X,x)\right)$, i.e. that the inclusion

$$\mathrm{Repf}_k\left(\pi_1^D(X,x)\right) \hookrightarrow \mathrm{Repf}_k(\Pi_0)$$

is actually an equivalence, which is what we wanted to show. ∎

4.4.4 Remark. Taking $X = \overline{X}$ in Corollary 4.4.3, we recover dos Santos' theorem [dS07, Prop. 13]. □

4.5 Finite stratified bundles restricted to curves

We continue to denote by k an algebraically closed field of characteristic $p > 0$. Let X be a smooth, separated, finite type k-scheme. In Section 3.4 we discussed the notion "curve-r.s." for stratified bundles on X. In this section we discuss the analogous notion relative to a fixed good partial compactification, and we show that for stratified bundles with finite monodromy we indeed have that $(X,\overline{X})$-regular singularity is equivalent to regular singularity on curves, in an appropriate sense.

We generalize the results of this section to the absolute notion of regular singularity from Definition 3.3.1 in Section 4.6.

4.5.1 Theorem. *Let $\overline{X}$ be a smooth, connected, separated, finite type k-scheme, and $(X,\overline{X})$ a good partial compactification with $D := \overline{X} \setminus X$. Then a finite stratified bundle E on X is $(X,\overline{X})$-regular singular, if and only if for every regular k-curve $\overline{C}$ and every morphism $\bar{\phi} : \overline{C} \to \overline{X}$, with $\bar{\phi}(\overline{C}) \not\subseteq D$, writing $C := \bar{\phi}^{-1}(X)$ and $\phi := \bar{\phi}|_C$, the stratified bundle ϕ^*E is $(C,\overline{C})$-regular singular.* □

Proof. If E is $(X,\overline{X})$-regular singular, the assumption that the image of $\overline{C}$ is not contained in the boundary divisor D, implies that Proposition 3.2.17 applies, so ϕ^*E is $(C,\overline{C})$-regular singular for every $\bar{\phi}$ from the statement.

Conversely, assume that ϕ^*E is regular singular for all $\bar{\phi}$ as above. Let ω be a k-valued fiber functor for $\langle E\rangle_{\otimes}$, then $\pi_1(\langle E\rangle_{\otimes},\omega)$ is a finite constant k-group scheme, because k is algebraically closed, and because $\pi_1(\langle E\rangle_{\otimes},\omega)$ is étale by Proposition 3.1.8. Write G for the finite group associated with the constant group scheme $\pi_1(\langle E\rangle_{\otimes},\omega)$. Let $h_{E,\omega} : X_{E,\omega} \to X$ be the trivializing torsor associated with ω, e.g. as in Proposition 4.1.8. It is galois étale with group G. The goal is to show that $h_{E,\omega}$ is tamely ramified with respect to D since Theorem 4.4.1 then implies that E is regular singular.

Consider the fiber product $f_{C,E} : X_{E,\omega} \times_X C \to C$. This is a finite étale covering, and it is the disjoint union $\coprod_j C_j \to C$, with C_j connected regular curves. Moreover, all C_j are C-isomorphic, say to $f : C' \to C$, as they are permuted by the action of G. To summarize notation, we have the following commutative diagram

$$\begin{tikzcd} C' \arrow[r, hook, "\phi"] \arrow[dr, "f"'] & X_{E,\omega} \times_X C = \coprod C' \arrow[r, "\mathrm{pr}"] \arrow[d] & X_{E,\omega} \arrow[d, "h_{E,\omega}"] \\ & C \arrow[r, "\phi"] & X \end{tikzcd} \tag{4.3}$$

Define $\omega_\phi : \langle \phi^* E \rangle_\otimes \to \mathrm{Vectf}_k$ by $F \mapsto H^0_{\mathrm{Strat}}(C', f^* F) = H^0(\mathrm{Strat}(C'), f^* F)$, see Definition C.1.13. This is a k-linear fiber functor since $f^* F$ is trivial for every $F \in \langle \phi^* E \rangle_\otimes$ (because $f^* \phi^* E$ is trivial by construction), and we obtain a commutative diagram

$$\begin{tikzcd} \langle E \rangle_\otimes \arrow[rr, "\phi^*"] \arrow[dr, "\omega"'] & & \langle \phi^* E \rangle_\otimes \arrow[dl, "\omega_\phi"] \\ & \mathrm{Vectf}_k & \end{tikzcd}$$

Indeed, by Proposition 4.1.4, (e), we know that (with notations from (4.3))

$$\begin{aligned} \omega(N) &= H^0(\mathrm{Strat}(X_{E,\omega}), \underbrace{h^*_{E,\omega} N}_{\text{trivial}}) \\ &= H^0(\mathrm{Strat}(C'), \phi^* \,\mathrm{pr}^*\, h^*_{E,\omega} N) \\ &= H^0(\mathrm{Strat}(C'), f^* \phi^* N) \\ &= \omega_\phi(\phi^* N) \end{aligned}$$

Thus we get a morphism of group schemes $\psi : \pi_1(\langle \phi^* E \rangle_\otimes, \omega_\phi) \to \pi_1(\langle E \rangle, \omega)$. Since $\phi^* E$ is finite, this is a morphism of constant k-group schemes, and by Proposition C.2.1 every object of $\langle \phi^* E \rangle_\otimes \subseteq \mathrm{Strat}(C)$ is a subquotient of $\phi^* E^n$ for some n. Thus by Proposition C.2.3, ψ is a closed immersion, $f : C' \to C$ is a $\pi_1(\langle \phi^* E \rangle, \omega_\phi)$-torsor; in fact it is the universal trivializing torsor for $\phi^* E$ and ω_ϕ, according to the following elementary lemma:

4.5.2 Lemma. *Let $H \subseteq G$ be finite groups, and $R \subseteq G$ be a set of representatives for G/H. Then $k[G] = \bigoplus_{r \in R} k[H]$ in $\mathrm{Repf}_k H$.* □

Theorem 4.4.1 then shows that f is tame with respect to $\overline{C} \setminus C$, since by assumption $\phi^* E$ is $(C, \overline{C})$-regular singular. As a disjoint union of copies of f, the covering $X_{E,\omega} \times_X C \to C$ then is also tame with respect to $\overline{C} \setminus C$. Finally, we invoke the theorem of Kerz-Schmidt Proposition B.1.8, which shows that $X_{E,\omega} \to X$ is tame with respect to $\overline{X} \setminus X$. ■

4.5.3 Corollary. *Let $\overline{X}$ be a smooth,* proper*, connected, finite type k-scheme, and $X \subseteq \overline{X}$ an open subscheme such that $D := \overline{X} \setminus X$ is a strict normal crossings divisor. Then a finite stratified bundle E on X is $(X, \overline{X})$-regular singular, if and only if $\phi^* E$ is $(C, \overline{C})$-regular singular for every morphism $\phi : C \to X$, with C a regular curve, and $\overline{C}$ the regular compactification of C.* □

Proof. This follows directly from Theorem 4.5.1, because any morphism ϕ extends uniquely to a morphism $\bar{\phi} : \overline{C} \to \overline{X}$ by the properness of $\overline{X}$. ∎

4.6 Finite regular singular stratified bundles in general

In this section we generalize the results about finite bundles which are regular singular with respect to a fixed good partial compactification to the general notion of regular singularities form Definition 3.3.1.

We start with a generalization of Proposition 4.2.1.

4.6.1 Proposition. *Let X be a smooth, separated, finite type k-scheme and $E \in \mathrm{Strat}^{\mathrm{rs}}(X)$. If E is finite then the elements of* $\mathrm{Exp}(E)$ *are torsion in $\mathbb{Z}_p/\mathbb{Z}$.* □

Proof. If E is finite, then E is trivialized on a finite étale covering $f : Y \to X$. Let $(X, \overline{X})$ be a good partial compactification of X. Then by Proposition 4.2.1, the exponents of all geometric discrete valuations ν which are realized by $(X, \overline{X})$ are torsion. This is true for every good partial compactification, so Proposition 3.3.12 shows that $\mathrm{Exp}(E)$ is torsion in $\mathbb{Z}_p/\mathbb{Z}$. ∎

The main result of this section is a generalization of Theorem 4.4.1:

4.6.2 Theorem. *Let X be a smooth, separated, finite type k scheme and E a stratified bundle on X. Then the following are equivalent:*

(a) *E is finite and regular singular.*

(b) *There is a finite tamely ramified* galois *étale morphism $f : Y \to X$, such that $f^*E \in \mathrm{Strat}(Y)$ is trivial.*

(c) *There is a finite tamely ramified étale morphism $f : Y \to X$, such that $f^*E \in \mathrm{Strat}(Y)$ is trivial.* □

Proof. Assume (a) and fix a fiber functor ω for $\langle E \rangle_{\otimes}$. Since E is $(X, \overline{X})$-regular singular for every good partial compactification $(X, \overline{X})$, Corollary 4.4.2 shows that the trivializing torsor $h_{E,\omega} : X_{E,\omega} \to X$ is tamely ramified with respect to every good partial compactification $(X, \overline{X})$, which is equivalent to $h_{E,\omega}$ being tame, see Definition B.2.1.

The rest is easy: (b) trivially implies (c), and if (c) holds, then E finite by Proposition 4.1.8, and it is a substratified bundle of the regular singular bundle $f_* \mathcal{O}_Y^{\operatorname{rank} E}$ (Proposition 3.3.6), so E is regular singular itself by Proposition 3.3.4. ∎

Again, the proof of the theorem shows more:

4.6.3 Corollary. *With the notation from Theorem 4.6.2, the following are equivalent:*

(a) *E is finite and regular singular,*

(b) *If ω is any neutral fiber functor for $\langle E \rangle_{\otimes}$, then the trivializing torsor $h_{E,\omega} : X_{E,\omega} \to X$ is finite étale and tamely ramified.*

Moreover, if one of the above holds, then if $f : Y \to X$ is any finite galois covering, such that f^*E is trivial, and $h : Y' \to X$ the maximal tame subcovering, then h^*E is trivial. □

PROOF. The proof of Corollary 4.4.2 holds without change. ■

We also obtain a generalization of Corollary 4.4.3.

4.6.4 Corollary. *Let X be a smooth, connected, separated k-scheme of finite type, and $x \in X(k)$ a rational point. Write $\Pi_X^{\mathrm{rs}} := \pi_1(\mathrm{Strat}^{\mathrm{rs}}(X), \omega_x)$, for the fiber functor ω_x associated to x. Then there is a quotient map of k-group schemes*

$$\phi : \Pi_X^{\mathrm{rs}} \twoheadrightarrow \pi_1^{\mathrm{tame}}(X, x),$$

where $\pi_1^{\mathrm{tame}}(X, x)$ is the constant group scheme associated to the profinite group classifying tame étale coverings of X, see Section B.2. This map factors uniquely through the canonical quotient $\Pi_X^{\mathrm{rs}} \twoheadrightarrow \pi_0(\Pi_X^{\mathrm{rs}})$, and induces a continuous isomorphism

$$\pi_0(\Pi_X^{\mathrm{rs}}) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}(X, x).$$ □

PROOF. First, note that $\pi_0(\Pi_X^{\mathrm{rs}})$ is (the constant k-group scheme associated with) a profinite group, because it is a quotient of $\pi_0(\pi_1(\mathrm{Strat}(X), \omega_x))$, which is the constant group scheme associated to $\pi_1(X, x)$, Remark 4.4.4.

By Corollary A.5 we know that the partially ordered set $\mathrm{PC}(X)$ of equivalence classes of good partial compactifications of X is directed. Also, with the notation from Corollary 4.4.3: $\Pi^{\mathrm{rs}}_{(X,\overline{X})} = \Pi^{\mathrm{rs}}_{(X,\overline{X}')}$ if $(X, \overline{X})$ and $(X, \overline{X}')$ are equivalent, so we can write Π_D^{rs} for $D \in \mathrm{PC}(X)$ an equivalence class of good partial compactifications. Similarly we can write $\pi_1^D(X, x)$ for the profinite group classifying finite étale coverings tamely ramified with respect to D, see Proposition B.1.5. By Corollary 4.4.3 we have an canonical continuous isomorphism

$$\pi_0(\Pi_D^{\mathrm{rs}}) \xrightarrow{\cong} \pi_1^D(X, x). \tag{4.4}$$

Moreover, if $D_1, D_2 \in \mathrm{PC}(X)$ and $D_2 \le D_1$, then there are canonical continuous quotient maps

$$\pi_0(\Pi_{D_2}^{\mathrm{rs}}) \twoheadrightarrow \pi_0(\Pi_{D_1}^{\mathrm{rs}}), \tag{4.5}$$

and

$$\pi_1^{D_2}(X, x) \twoheadrightarrow \pi_1^{D_1}(X, x). \tag{4.6}$$

such that (4.5) induces (4.6) via the isomorphism from Corollary 4.4.3. This means that the inductive systems $(\pi_0(\Pi_D^{\mathrm{rs}}))_{D \in \mathrm{PC}(X)}$ and $(\pi_1^D(X, x))_{D \in \mathrm{PC}(X)}$ with the transition morphisms (4.5) and (4.6) are isomorphic.

Next, by Proposition B.2.5 we know that that $\pi_1^{\mathrm{tame}}(X, x) = \varinjlim_D \pi_1^D(X, x)$ in the category of profinite groups. Likewise, we show that

$$\pi_0(\Pi_X^{\mathrm{rs}}) = \varinjlim_{D \in \mathrm{PC}(X)} \pi_0(\Pi_D^{\mathrm{rs}}).$$

Indeed, a profinite group G and a compatible system of continuous maps

$$(\pi_0(\Pi_D^{\mathrm{rs}}) \to G)_{D \in \mathrm{PC}(X)},$$

gives rise (via the isomorphism (4.4)) to a compatible system of continuous maps

$$(\phi_D : \pi_1^D(X,x) \twoheadrightarrow G)_{D\in \mathrm{PC}(X)}$$

and thus to a tame pro-étale galois covering $Y \to X$ with group $\mathrm{im}(\phi_D)$. The group $\mathrm{im}(\phi_D)$ is profinite and independent of D, since the transition morphisms of the inductive system are surjective. If $\mathrm{im}(\phi_D) = \varprojlim_N \mathrm{im}(\phi_D)/N$ with $N \subseteq \mathrm{im}(\phi_D)$ normal open subgroups, then $Y = \varprojlim_N Y_N$, and each $f_N : Y_N \to X$ corresponds to a regular singular bundle $(f_N)_*\mathcal{O}_{Y_N}$ on X. We obtain a continuous map

$$\pi_0(\Pi_X^{\mathrm{rs}}) \twoheadrightarrow \varprojlim_N \pi_1(\langle (f_N)_*\mathcal{O}_{Y_N}\rangle_{\otimes}, \omega_x) = \varprojlim_N \mathrm{im}(\phi_D)/N = \mathrm{im}(\phi_D) \hookrightarrow G$$

Conversely, any continuous map $\pi_0(\Pi_X^{\mathrm{rs}}) \to G$ gives a compatible system of continuous maps

$$(\pi_0(\Pi_D^{\mathrm{rs}}) \twoheadrightarrow \pi_0(\Pi_X^{\mathrm{rs}}) \to G)_{D\in \mathrm{PC}(X)},$$

and the two constructions are mutually inverse. Thus we have proven:

$$\pi_0(\Pi_X^{\mathrm{rs}}) = \varinjlim_D \pi_0(\Pi_D^{\mathrm{rs}})) \cong \varinjlim_D \pi_1^D(X,x) = \pi_1^{\mathrm{tame}}(X,x).$$

■

Finally we derive a generalization of Corollary 4.5.3:

4.6.5 Theorem. *Let X be a smooth, finite type k-scheme, Then a* finite *stratified bundle E on X is regular singular, if and only if ϕ^*E is regular singular for every morphism $\phi : C \to X$, with C a regular curve.* □

PROOF. Let ω be a neutral fiber functor for $\langle E\rangle_{\otimes}$, and let $h_{E,\omega} : X_{E,\omega} \to X$ be the trivializing torsor for E and ω. By Corollary 4.6.3, E is regular singular if and only if $h_{E,\omega}$ is tamely ramified. By Theorem B.2.2 of Kerz-Schmidt, $h_{E,\omega}$ is tame if and only if $h_{E,\omega} \times_X \phi : X_{E,\omega} \times_X C \to C$ is tame for all $\phi : C \to X$ as in the claim.

As in the proof of Theorem 4.5.1 it follows that $X_{E,\omega} \times_X C$ is C-isomorphic to a disjoint union of copies of the trivializing torsor $h_{\phi^*E,\omega_\phi} : C_{\phi^*E,\omega_\phi} \to C$, where ω_ϕ is the fiber functor as defined in the proof of Theorem 4.5.1. So $h_{E,\omega} \times_X \phi$ is tame if and only if h_{ϕ^*E,ω_ϕ} is tame. But this is equivalent to ϕ^*E being regular singular, again by Corollary 4.6.3. This completes the proof. ■

Chapter 5

Regular singular stratified bundles on simply connected varieties

In this chapter we start exploring an analog of a consequence of the Riemann-Hilbert correspondence, as developed by P. Deligne in [Del70]: If $\mathbb{C}$ is an algebraically closed field of characteristic 0, X a smooth, separated, finite type $\mathbb{C}$-scheme and $x \in X(\mathbb{C})$ a rational point, then there is an equivalence of categories

$$\mathrm{Repf}_{\mathbb{C}}(\pi_1^{\mathrm{top}}(X(\mathbb{C}), x)) \to \mathrm{Strat}^{\mathrm{rs}}(X).$$

Here usually $\mathrm{Strat}^{\mathrm{rs}}(X)$ is identified with the category of coherent $\mathcal{O}_X$-modules with flat, regular singular connection, and $\pi_1^{\mathrm{top}}(X(\mathbb{C}), x)$ denotes the fundamental group of the topological space $X(\mathbb{C})$ with its compex topology. From a tannakian point of view, if $\omega_x : \mathrm{Strat}^{\mathrm{rs}}(X) \to \mathrm{Vectf}_{\mathbb{C}}$ is the neutral fiber functor associated with x, this shows that there is an isomorphism of pro-algebraic $\mathbb{C}$-group schemes

$$\pi_1(\mathrm{Strat}(X), \omega_x) \cong \left(\pi_1^{\mathrm{top}}(X(\mathbb{C}), x)\right)^{\mathrm{alg}}$$

where for an abstract group G, the $\mathbb{C}$-group scheme G^{alg} is the *algebraic hull of* G, i.e. the pro-algebraic $\mathbb{C}$-group scheme associated with $\mathrm{Repf}_{\mathbb{C}}\, G$ via tannaka duality.

A theorem of Malcev [Mal40] (rediscovered and applied by Grothendieck in [Gro70] in a context very similar to ours) states that for two abstract, finitely generated groups G, H, a homomorphism $f : G \to H$ induces an isomorphism $f^{\mathrm{alg}} : G^{\mathrm{alg}} \xrightarrow{\cong} H^{\mathrm{alg}}$, if f induces an isomorphism $\hat{f} : \widehat{G} \xrightarrow{\cong} \widehat{H}$ of the profinite completions of G and H.

In our situation there is a canonical isomorphism

$$\pi_1^{\mathrm{top}}\widehat{(X(\mathbb{C}), x)} \cong \pi_1^{\text{ét}}(X, x),$$

see [SGA1, Exp. XII]. Thus if $\pi_1^{\text{ét}}(X, x) = 0$, then $(\pi_1^{\mathrm{top}}(X(\mathbb{C}), x))^{\mathrm{alg}} = 0$. Conversely, every nontrivial finite covering of X gives rise to a nontrivial regular singular connection on X. We obtain:

5.0.6 Theorem (Riemann-Hilbert, Deligne, Malcev, Grothendieck). *If X is a smooth, connected, separated, finite type $\mathbb{C}$-scheme and $x \in X(\mathbb{C})$ a rational point, then $\pi_1^{\text{ét}}(X,x) = 0$ if and only if every regular singular stratified bundle on X is trivial.* □

The goal of this chapter is to start an analysis of similar statements in the context of Chapter 3.

5.0.7 Example. The following is known over an algebraically closed field k of characteristic $p > 0$:

(a) In the case that X is smooth, connected, *projective*, of finite type over k, then every stratified bundle on X is regular singular, and it was conjectured by Gieseker in [Gie75] that $\pi_1^{\text{ét}}(X,\bar{x}) = 0$ holds if and only if every stratified bundle on X is trivial. This was proved by H. Esnault and V. Mehta, [EM10].

(b) It follows directly from Corollary 4.6.4 that $\pi_1^{\text{tame}}(X,x) = 0$ for some $x \in X(k)$, if every regular singular stratified bundle is trivial. Conversely, if $\pi_1^{\text{tame}}(X,x) = 0$, then Corollary 4.6.4 shows that every nontrivial regular singular stratified bundle E on X has a *connected* monodromy group.

(c) If $X = \mathbb{A}^n_k$, then there are no nontrivial regular singular stratified bundles on X, see e.g. [Esn12, Rem. 4.4]: By [Gie75, Thm. 5.3], it follows that every regular singular stratified bundle is a direct sum of rank 1 stratified bundles.

Let $D = \mathbb{P}^n_k \setminus \mathbb{A}^n_k$. We show that a $\mathscr{D}_{\mathbb{P}^n_k/k}(\log D)$-action on a line bundle $\mathcal{O}_{\mathbb{P}^n_k}(mD)$, $m > 0$, automatically extends to a $\mathscr{D}_{\mathbb{P}^n_k/k}$-action and thus is trivial. Indeed, $\mathcal{O}_{\mathbb{P}^n_k}(mD)$ can only carry the canonical logarithmic connection d given by the inclusion $\mathcal{O}_{\mathbb{P}^n} \subseteq \mathcal{O}_{\mathbb{P}^n_k}(mD)$: Any logarithmic connection on $\mathcal{O}_{\mathbb{P}^n_k}(mD)$ is of the form $\nabla = d + \alpha$, for $\alpha \in H^0(\mathbb{P}^n_k, \Omega^1_{\mathbb{P}^n_k}(\log D)) = 0$, because giving a logarithmic connection is equivalent to giving a splitting of the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n_k} \otimes \mathcal{O}_{\mathbb{P}^n_k}(mD) \to \mathcal{P}^1_{\mathbb{P}^n_k/k}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n_k}(mD) \to \mathcal{O}_{\mathbb{P}^n_k}(mD) \to 0.$$

Thus the first step of the logarithmic stratification on $\mathcal{O}_{\mathbb{P}^n_k}(mD)$ is in fact regular and we can descend to the frobenius pullback $(\mathbb{P}^n_k)^{(1)}$ and $\mathcal{O}^{\nabla}_{\mathbb{P}^n_k}$. Here we reapply the argument to see that the second step of the logarithmic stratification is also regular, etc.

(d) In Section 5.1 we prove that $\mathrm{Strat}^{\mathrm{rs}}(X)$ does not contain any rank 1 objects if $\pi_1^{\text{tame}}(X,x) = 0$, extending an argument of [EM10]. Note that by Example 3.3.9 every rank 1 stratified bundle is automatically regular singular. □

These results can be considered as evidence for the existence of a positive answer to the following question:

5.0.8 Question. *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth, separated, finite type k-scheme and $x \in X(k)$ a rational point. Is it true that $\pi_1^{\text{tame}}(X,\bar{x}) = 0$ if and only if every regular singular stratified bundle on X is trivial, i.e. if and only if $\pi_1(\mathrm{Strat}^{\mathrm{rs}}(X), \omega_x) = 0$, where $\omega_x : \mathrm{Strat}^{\mathrm{rs}}(X) \to \mathrm{Vectf}_k$ is the fiber functor associated with x?* □

In fact even the following weaker question is interesting:

5.0.9 Question. *With the notations from Question 5.0.8, is it true that there are non nontrivial stratified bundles if $\pi_1^{\text{ét}}(X,x)=0$?* $\square$

Note that the condition $\pi_1^{\text{ét}}(X,x)=0$ is much stronger condition than requiring $\pi_1^{\text{tame}}(X,x)=0$; for example it implies that X can only have constant global functions, see Proposition 5.1.2.

According to Proposition 3.3.8, if X is an open subscheme of a smooth finite type k-scheme $\overline{X}$ such that $\overline{X}\setminus X$ has codimension ≥ 2 in $\overline{X}$, then the restriction functor $\mathrm{Strat}^{\mathrm{rs}}(\overline{X})\to\mathrm{Strat}^{\mathrm{rs}}(X)$ is an equivalence. Similarly, $\pi_1^{\text{tame}}(X,x)=\pi_1^{\text{tame}}(\overline{X},x)$, and if $\overline{X}$ also happens to be projective then $\mathrm{Strat}(\overline{X})=\mathrm{Strat}^{\mathrm{rs}}(\overline{X})$ and $\pi_1^{\text{tame}}(\overline{X},x)=\pi_1(\overline{X},x)$. This means that for such X Question 5.0.8 reduces to the projective case already solved by H. Esnault and V. Mehta, [EM10].

Nevertheless, it is not hard to construct smooth, separated, finite type k-schemes X with $\pi_1(X,x)=0$ such that X is not embeddable in a smooth projective variety such that the boundary has codimension ≥ 2:

5.0.10 Example. Blow up $\mathbb{P}^2_k$ in a closed point P to get a regular surface $\overline{X}_1$ with exceptional divisor E_1. Then $E_1\cong\mathbb{P}^1_k$, and $E_1^2=-1$. Next, blow up $\overline{X}_1$ in a closed point of E_1 to get $\overline{X}_2$ with exceptional divisor E_2, and let $\tilde{E}_1$ be the proper transform of E_1 in $\overline{X}_2$. Then we have $\tilde{E}_1^2=-2$: Indeed, the projection formula implies that

$$-1=E_1^2=\tilde{E}_1\cdot(E_2+\tilde{E}_1)=\tilde{E}_1\cdot E_2+\tilde{E}_1^2,$$

and $\tilde{E}_1\cdot E_2=1$, since Q has multiplicity 1 on E_1 (see e.g. [Har77, Cor. 3.7]).

Thus, if $X_2:=\overline{X}_2\setminus\tilde{E}_1$, and X_2' any compactification of X_2 such that $X_2'\setminus X_2$ has codimension ≥ 2, then X_2' is singular: The curve $\tilde{E}_1$ in $\overline{X}_2$ can be contracted, but only to a singular point, since $\tilde{E}_1^2=-2$, and X_2' is a modification of this contraction of $\overline{X}_2$.

Finally, since X_2 contains a dense open subset isomorphic to $\mathbb{P}^2_k\setminus\{P\}$, it follows that $\pi_1(X_2,x)=\pi_1^{\text{tame}}(X_2,x)=0$. $\square$

5.1 Stratified bundles of rank 1

As usual, let k denote an algebraically closed field of characteristic $p>0$. The goal of this section is to prove the following theorem:

5.1.1 Theorem. *If X is a smooth, connected k-scheme of finite type such that the maximal abelian pro-prime-to-p quotient $\pi_1(X)^{ab,(p')}$ of $\pi_1(X,\bar{x})$ is trivial for some geometric point $\bar{x}$ of X, then every stratified line bundle on X is trivial.* $\square$

Note that every covering of degree prime to p is tame, so the condition of the theorem is in particular fulfilled if $\pi_1^{\text{tame}}(X,\bar{x})=0$. Also note that a stratified line bundle on X is always regular singular by Example 3.3.9, so Theorem 5.1.1 really implies the formulation of Example 5.0.7, (d).

Recall that fundamental groups of connected schemes depend on the choice of the geometric base point only up to inner automorphism. Thus, whenever we consider abelian quotients, we drop the geometric base point from the notation.

Before we start to prove Theorem 5.1.1, we need to know a fact about global functions on simply connected varieties:

5.1.2 Proposition. *If X is a connected normal k-scheme of finite type, such that the maximal abelian pro-ℓ-quotient $\pi_1(X)^{ab,(\ell)}$ is trivial for some $\ell \neq p$, then $H^0(X, \mathcal{O}_X^\times) = k^\times$. If k has positive characteristic p, and $\pi_1(X)^{ab,(p)} = 1$, then $H^0(X, \mathcal{O}_X) = k$.* □

PROOF. The neat argument for the first assertion is due to Hélène Esnault. Assume $f \in H^0(X, \mathcal{O}_X^\times) \setminus k^\times$. Then f induces a dominant morphism $f' : X \to \mathbb{G}_{m,k} \cong \mathbb{A}^1_k \setminus \{0\}$, as f' is given by the map $k[x^{\pm 1}] \to H^0(X, \mathcal{O}_X)$, $x \mapsto f$, which is injective if, and only if, f is transcendental over k. Thus f' induces an *open* morphism $\pi_1(X) \to \pi_1(\mathbb{G}_{m,k})$, see e.g. [Sti02, Lemma 4.2.10]. But under our assumption, the maximal abelian pro-ℓ-quotient of the image of this morphism is trivial, so the image of $\pi_1(X)$ cannot have finite index in the group $\pi_1(\mathbb{G}_{m,k})$, as in fact $\pi_1(\mathbb{G}_{m,k})^{(\ell)} \cong \widehat{\mathbb{Z}}^{(\ell)} = \mathbb{Z}_\ell$.

For the second assertion, if $f \in H^0(X, \mathcal{O}_X) \setminus k$, then by the same arguments as above, f induces a dominant morphism $X \to \mathbb{A}^1_k$, and hence an open map $\pi_1(X) \to \pi_1(\mathbb{A}^1_k)$. For k of positive characteristic it is known that $\pi_1(\mathbb{A}^1_k)$ has an infinite maximal pro-p-quotient; in fact it is a free pro-p-group of infinite rank (by [Kat86, 1.4.3, 1.4.4] we have $H^2(\pi_1(X), \mathbb{F}_p) = 0$, so $\pi_1(X)^{(p)}$ is free pro-p of rank $\dim_{\mathbb{F}_p} H^1(\mathbb{A}^1_k, \mathbb{F}_p) = \#k$). Thus the image of $\pi_1(X)$ in this group can only have finite index, if $\pi_1(X)^{\mathrm{ab},(p)} \neq 1$. ■

5.1.3 Remark. Proposition 5.1.2 gives a proof of the well-known fact that over a field k of positive characteristic, unlike in characteristic 0, no affine k-scheme of positive dimension is simply connected. □

5.1.4 Corollary. *Let X be a smooth, connected k-scheme of finite type and $L \in \mathrm{Strat}(X)$ a stratified line bundle. Using Theorem 3.1.9, we identify L with the datum $(L_n, \sigma_n)_{n\geq 0}$, where L_n is a line bundle on X and $\sigma_n : F^*L_{n+1} \xrightarrow{\cong} L_n$ an $\mathcal{O}_X$-linear isomorphism.*

If $\pi_1(X)^{ab,(\ell)} = 1$ for some prime $\ell \neq p$, then the isomorphism class of L is uniquely determined by the isomorphism classes of the L_n. □

PROOF. This follows from Proposition 5.1.2, and the argument is essentially contained in the proof of [Gie75, Prop. 1.7]: Let $M := (M_n, \tau_n)$ be a second stratified line bundle on X and $u_n : L_n \to M_n$ isomorphisms of $\mathcal{O}_X$-modules. We will construct an isomorphism of stratified line bundles $L \to M$. Consider the following diagram:

$$\begin{tikzcd} L_0 \arrow[r, "\sigma_0"] \arrow[d, "u_0"', "\cong"] & F^*L_1 \\ M_0 \arrow[r, "\tau_0"] & F^*M_1 \end{tikzcd}$$

The automorphism $\lambda := \tau_0 u_0 \sigma_0^{-1} F^*(u_1^{-1})$ of F^*M_1 corresponds to a global unit $\lambda \in \Gamma(X, \mathcal{O}_X^\times)$. By Proposition 5.1.2, X has only constant global units, so there is a p-th root $\lambda^{1/p}$ of λ, which defines an automorphism of M_1 such that $F^*\lambda^{1/p} = \lambda$. Defining $f_0 := u_0$ and $f_1 := \lambda^{1/p} u_1$ gives the first two steps of defining an isomorphism of stratified bundles $f : L \to M$. We can continue this process. ■

The proof of Theorem 5.1.1 relies on the following theorem, which will be proved in Section 5.1.2:

5.1.5 Theorem. *Let k be an algebraically closed field of characteristic $p \geq 0$. If X is a connected, regular, separated k-scheme of finite type, and if the maximal abelian pro-ℓ quotient $\pi_1(X)^{ab,(\ell)}$ is trivial for some $\ell \neq p$, then* $\operatorname{Pic} X$ *is a finitely generated abelian group.* □

If we admit the truth of Theorem 5.1.5 for now, we can give a short proof of Theorem 5.1.1:

PROOF (OF THEOREM 5.1.1.). This is an adaptation of an argument from the introduction of [EM10]. Let $L = (L_n, \sigma_n)_{n\geq 0}$ be a stratified line bundle on X. By Corollary 5.1.4 we only need to show that the classes of L_n in $\operatorname{Pic} X$ are all trivial. Note that L_n is infinitely p-divisible in $\operatorname{Pic} X$ for all n. For smooth X as in the assertion, we know from Theorem 5.1.5 that $\operatorname{Pic} X$ is finitely generated. Since a nontrivial element of a finitely generated abelian group is infinitely p-divisible if and only if it has finite order prime to p, it follows that L_n is torsion of order prime to p in $\operatorname{Pic} X$. By Kummer theory we know that

$$\operatorname{Pic}(X)[m] = H^1_{\text{ét}}(X, \mathbb{Z}/m\mathbb{Z}(1)) = \operatorname{Hom}_{\text{cont}}(\pi_1^{\text{ab},(p')}(X), \mathbb{Z}/m\mathbb{Z}) = 0$$

for all $m \in \mathbb{N}$ with $(m,p) = 1$. This shows that $\operatorname{Pic} X$ does not have nontrivial prime-to-p torsion, so $L_n = \mathcal{O}_X$. ■

5.1.6 Remark. The difficulty in proving Theorem 5.1.5 is the fact that resolution of singularities is not available in positive characteristic. Let $\ell \neq p$ be a prime and lets assume that X is a smooth, separated, finite type k-scheme with $\pi_1^{\text{ab},(\ell)}(X) = 0$, and that there is a *regular, proper*, finite type k-scheme $\overline{X}$ and a dominant open immersion $j : X \hookrightarrow \overline{X}$. Then by the regularity assumption the map $j^* : \operatorname{Pic} \overline{X} \twoheadrightarrow \operatorname{Pic} X$ is surjective, so to prove that $\operatorname{Pic} X$ is finitely generated, it suffices to prove that $\operatorname{Pic} \overline{X}$ is finitely generated. The induced map $\pi_1^{\text{ab},(\ell)}(X) \twoheadrightarrow \pi_1^{\text{ab},(\ell)}(\overline{X})$ is also surjective. Since $\overline{X}$ is proper, the relative Picard functor for $\overline{X}/k$ is representable by a k-group scheme $\operatorname{Pic}_{\overline{X}/k}$, locally of finite type over k. The connected component of the origin with its reduced structure $\operatorname{Pic}^{0,\text{red}}_{\overline{X}/k}$ is an abelian variety and by Kummer theory we can compute the torsion subgroup $\operatorname{Pic}^{0,\text{red}}_{\overline{X}/k}[\ell^n] \subseteq \operatorname{Pic} \overline{X}$:

$$\operatorname{Pic}^{0,\text{red}}_{\overline{X}/k}[\ell^n] = \operatorname{Hom}_{\text{cont}}(\pi_1^{\text{ab},(\ell)}(\overline{X}), \mathbb{Z}/\ell^n) = 1.$$

This implies that the abelian variety $\operatorname{Pic}^{0,\text{red}}_{\overline{X}/k}$ is trivial, since for an abelian variety A of dimension g, $A[\ell^n] = (\mathbb{Z}/\ell^n)^{2g}$. Thus $\operatorname{Pic} \overline{X} = \operatorname{Pic}_{\overline{X}/k}(k) = \operatorname{NS}(\overline{X})$ is the Néron-Severi group of $\overline{X}$, which is finitely generated by [SGA6, Thm. XIII.5.1].

The strategy of the proof of Theorem 5.1.5 in the general case is to use de Jong's theorem on alterations to replace X by a simplicial scheme, which admits a "smooth compactification" in a suitable sense. We then mimic the argument from the previous paragraph for simplicial line bundles and the simplicial Picard group. □

5.1.1 Simplicial Picard groups

For background on the simplicial techniques used in this section, we refer to [Del74].

In [BVS01, 4.1] and [Ram01, 3.1], the simplicial Picard group and the simplicial Picard functor are defined as follows:

5.1.7 Definition. Let S be a scheme. If $\delta^i_k : X_i \to X_{i-1}$ denote the face maps of a simplicial S-scheme $X_\bullet$, then $\mathrm{Pic}(X_\bullet)$ is defined to be the group of isomorphism classes of pairs (L, α) consisting of a line bundle L on X_0 and an isomorphism $\alpha : (\delta^1_0)^*L \to (\delta^1_1)^*L$ on X_1, satisfying cocycle condition $(\delta^2_2)^*(\alpha)(\delta^2_0)^*(\alpha) = (\delta^2_1)^*(\alpha)$ on X_2. The simplicial Picard functor $\mathrm{Pic}_{X_\bullet/S}$ is obtained by fpqc-sheafifying the functor $T \mapsto \mathrm{Pic}(X_\bullet \times_S T)$. □

It is not hard to check that $\mathrm{Pic}(X_\bullet)$ is canonically isomorphic to $\mathbb{H}^1(X_\bullet, \mathcal{O}^\times_{X_\bullet})$ and to the group of isomorphism classes of invertible $\mathcal{O}_{X_\bullet}$-modules, see [BVS01, A.3]. We will use the following representability and finiteness statements.

5.1.8 Theorem. *Let k be an algebraically closed field, X a proper k-scheme of finite type, and $X_\bullet$ a proper, simplicial k-scheme of finite type (which means that all the X_n are proper, and of finite type over k).*

(a) *The relative Picard functor associated with $X \to \operatorname{Spec} k$ is representable by a separated commutative group scheme $\mathrm{Pic}_{X/k}$, locally of finite type over k, which is the disjoint union of open, quasi-projective subschemes, see [SGA6, Cor. XII.1.2].*

(b) *The Néron-Severi group* $\mathrm{NS}(X) = \mathrm{Pic}_{X/k}(k)/\mathrm{Pic}^0_{X/k}(k)$ *is a finitely generated abelian group, see [SGA6, Thm. XIII.5.1].*

(c) *If X also* normal, *then the connected component $\mathrm{Pic}^0_{X/k}$ of the origin is projective, see [Kle05, Thm. 5.4, Rem. 5.6], so the reduced subscheme $\mathrm{Pic}^{0,\mathrm{red}}_{X/k}$ is an abelian variety.*

(d) *The simplicial Picard functor is representable by a group scheme $\mathrm{Pic}_{X_\bullet/k}$, locally of finite type over k, see [Ram01, Thm. 3.2].*

(e) *If X_n is reduced for all n, and X_0 normal, then the connected component $\mathrm{Pic}^{0,\mathrm{red}}_{X_\bullet/k}$ of the origin is semi-abelian, see [Ram01, Cor. 3.5].* □

Statement (b) from above can be generalized to the simplicial situation. In the case that $\mathrm{char}(k) = 0$, this is sketched in [BVRS09, Sec. 3].

5.1.9 Proposition. *Let k be an algebraically closed field. For a proper reduced simplicial k-scheme $X_\bullet$, with X_n of finite type for all n and X_0 normal, the* simplicial Néron-Severi group $\mathrm{NS}(X_\bullet) := \mathrm{Pic}_{X_\bullet/k}(k)/\mathrm{Pic}^0_{X_\bullet/k}(k)$ *is finitely generated.* □

PROOF. Let $\tau : X_\bullet \to \operatorname{Spec} k$ denote the structure morphism. The spectral sequence (see e.g. [Del74, (5.2.3.2)])

$$E_1^{p,q} = H^q(X_p, \mathcal{O}^\times_{X_p}) \Longrightarrow \mathbb{H}^{p+q}(X_\bullet, \mathcal{O}^\times_{X_\bullet})$$

gives rise to the exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow \mathbb{H}^1(X_\bullet, \mathcal{O}_{X_\bullet}) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0}$$

and thus, as in [Ram01, p. 284], after sheafifying we get an exact sequence of fpqc-sheaves (in fact group schemes by the representability of $\mathrm{Pic}_{X_\bullet/k}$)

$$0 \to T \to \mathrm{Pic}^{\mathrm{red}}_{X_\bullet/k} \to K \xrightarrow{d_2} W,$$

where $K := \ker(\delta_0^* - \delta_1^* : \mathrm{Pic}_{X_0/k} \to \mathrm{Pic}_{X_1/k})^{\mathrm{red}}$,

$$T := \frac{\ker((\tau_1)_*\mathbb{G}_{m,X_1} \xrightarrow{\delta_0^*-\delta_1^*+\delta_2^*} (\tau_2)_*\mathbb{G}_{m,X_2})}{\mathrm{im}((\tau_0)_*\mathbb{G}_{m,X_0} \xrightarrow{\delta_0^*-\delta_1^*} (\tau_1)_*\mathbb{G}_{m,X_1})},$$

and W is affine. The scheme T is an affine k-scheme with finitely many connected components, and a k-torus as neutral component (compare [Ram01, top of p. 284]). This follows from the fact that τ_n is proper and X_n reduced, since this implies that for every k-scheme S we have $\tau_{n,*}\mathbb{G}_{m,X_n}(S) = \mathcal{O}^{\times}_{X_n\times_k S}(X_n \times_k S) = \mathbb{G}_{m,k}(S)^{\pi_0(X_n)}$, see [Gro63, Prop. 7.8.6].

As X_0 is normal, $\mathrm{Pic}^{0,\mathrm{red}}_{X_0/k}$ is an abelian variety, and hence so is K^0. Since W is affine, any homomorphism $K^0 \to W$ is trivial, so $K^0 \subseteq \ker(d_2)^0$. But $\ker(d_2)^0 \subseteq K^0$, so we have equality. Moreover, $\mathrm{Pic}^{0,\mathrm{red}}_{X_\bullet/k}$ maps surjectively to K^0. This shows that the kernel of the map

$$\mathrm{Pic}_{X_\bullet/k}(k)/\mathrm{Pic}^0_{X_\bullet/k}(k) = \mathrm{NS}(X_\bullet) \to K(k)/K^0(k)$$

is

$$\frac{T(k)\,\mathrm{Pic}^0_{X_\bullet/k}(k)}{\mathrm{Pic}^0_{X_\bullet/k}(k)},$$

■

because, if L maps to $M \in K^0(k)$, then there is some $L^0 \in \mathrm{Pic}^0_{X_\bullet/k}(k)$ also mapping to M, and the difference is in $T(k)$.

As T has only finitely many connected components, this kernel is finite.

The group $K(k)/K^0(k)$ maps to the group of connected components $\mathrm{NS}(X_0)$ of $\mathrm{Pic}_{X_0/k}$. The kernel of this map is $(\mathrm{Pic}^0_{X_0/k}(k) \cap K(k))/K^0(k)$, which is finite, as $\mathrm{Pic}^0_{X_0/k} \cap K$ has only finitely many connected components, and the subgroup $K^0 \subseteq \mathrm{Pic}^0_{X_0/k}$ is the neutral component of $\mathrm{Pic}^0_{X_0/k} \cap K$. Hence $K(k)/K^0(k)$ is finitely generated, because $\mathrm{NS}(X_0)$ is finitely generated by Theorem 5.1.8. This shows that $\mathrm{NS}(X_\bullet)$ is finitely generated.

Recall that with an X-scheme X_0 one can associate an X-augmented simplicial scheme $\mathrm{cosk}_0(X_0)_\bullet$, the 0-*coskeleton*, defined by taking $\mathrm{cosk}_0(X_0)_n$ to be the n-fold fiber product of X_0 over X, with the necessary maps given by the various projections (resp. diagonals) to (resp. from) $\mathrm{cosk}_0(X_0)_{n-1}$. The 0-coskeleton has the following universal property: If $Y_\bullet$ is a simplicial scheme with augmentation to X, then there is a bifunctorial bijection $\mathrm{Hom}_X(Y_0, X_0) \cong \mathrm{Hom}_X(Y_\bullet, \mathrm{cosk}_0(X_0)_\bullet)$. In particular, if $X_\bullet$ is a simplicial scheme, then there is a unique simplicial X-morphism $\gamma : X_\bullet \to \mathrm{cosk}_0(X_0)_\bullet$, with $\gamma_0 = \mathrm{id}_{X_0}$. For the general construction see, e.g., [Del74, 5.1.1].

5.1.10 Lemma. *If $\tau : X_\bullet \to X$ is an augmented simplicial k-scheme such that X_n is separated and of finite type over k for all n, and $\gamma : X_\bullet \to \mathrm{cosk}_0(X_0)_\bullet$ the morphism of augmented simplicial schemes such that $\tau_0 = \mathrm{id}_{X_0}$, then the kernel of the induced morphism $\mathrm{Pic}((\mathrm{cosk}_0(X_0)_\bullet) \to \mathrm{Pic}\, X_\bullet$ is finitely generated.* □

PROOF. We have the following situation:

$$\gamma_\bullet : X_\bullet \longrightarrow \operatorname{cosk}_0(X)_\bullet$$

$$\begin{tikzcd}
\vdots & \vdots \\
X_1 \arrow[r, "\gamma_1"] & X_0' \\
X_0 \arrow[u] \arrow[d] \arrow[r, equal, "\gamma_0=\mathrm{id}"] & X_0 \arrow[u] \arrow[d] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

with the maps $\delta_i : X_1 \to X_0$ and $p_i : X_0' \to X_0$,

where $X_0' := X_0 \times_X X_0$ and p_i the projection to the i-th factor.

If $(L, \alpha : p_1^* L \xrightarrow{\cong} p_2^* L) \in \operatorname{Pic}(\operatorname{cosk}_0(X_0)_\bullet)$ pulls back to the trivial element in $\operatorname{Pic}(X_\bullet)$, then there is some isomorphism $\beta : L \to \mathcal{O}_{X_0}$, such that the diagram

$$\begin{tikzcd}
\delta_0^* L \arrow[r, "\gamma_1^*\alpha"] \arrow[d, "\delta_0^*\beta"'] & \delta_1^* L \arrow[d, "\delta_1^*\beta"] \\
\mathcal{O}_{X_1} \arrow[r, equal, "\mathrm{id}"] & \mathcal{O}_{X_1}
\end{tikzcd}$$

commutes, and $(p_2^*\beta)\alpha(p_1^*\beta)^{-1}$ is an automorphism of $\mathcal{O}_{X_0'}$, pulling back the identity on X_1. Hence $(p_2^*\beta)\alpha(p_1^*\beta)^{-1}$ is an element of

$$\ker(\gamma_1^* : \Gamma(X_0', \mathcal{O}_{X_0'}^\times) \longrightarrow \Gamma(X_1, \mathcal{O}_{X_1}^\times)).$$

Note that $(p_2^*\beta)\alpha(p_1^*\beta)^{-1} = 1$ if and only if (L, α) is trivial in $\operatorname{Pic}(\operatorname{cosk}_0(X_0)_\bullet)$.

Replacing β by $\beta\lambda$ for some $\lambda \in \ker(\delta_0^* - \delta_1^* : \Gamma(X_0, \mathcal{O}_{X_0}^\times) \to \Gamma(X_1, \mathcal{O}_{X_1}^\times))$ gives a new trivialization $\gamma^*(L, \alpha) \cong (\mathcal{O}_{X_0}, \mathrm{id})$, and any trivialization can be reached like this (trivializations of the line bundle L are a $\mathbb{G}_m$-torsor, and to get a trivialization of the *pair* $\gamma^*(L, \alpha) = (L, \gamma_1^*\alpha)$, the condition that $1 = (\delta_1^*\lambda)^{-1}(\delta_0^*\lambda)$ is necessary and sufficient). Next, observe that $p_1^* - p_2^*$ (or rather p_1^*/p_2^*) induces a map

$$\ker(\Gamma(X_0, \mathcal{O}_{X_0}^\times) \xrightarrow{\delta_0^* - \delta_1^*} \Gamma(X_1, \mathcal{O}_{X_1}^\times)) \to \ker(\Gamma(X_0', \mathcal{O}_{X_0'}^\times) \xrightarrow{\gamma_1^*} \Gamma(X_1, \mathcal{O}_{X_1}^\times)).$$

Putting all of this together, we see that we obtain an injective map

$$\ker\left(\operatorname{Pic}((\operatorname{cosk}_0 X_0)_\bullet) \to \operatorname{Pic}(X_\bullet)\right) \longrightarrow \frac{\ker(\Gamma(X_0', \mathcal{O}_{X_0'}^\times) \xrightarrow{\gamma_1^*} \Gamma(X_1, \mathcal{O}_{X_1}^\times))}{(p_1 - p_2)^*(\ker(\Gamma(X_0, \mathcal{O}_{X_0}^\times) \xrightarrow{\delta_0^* - \delta_1^*} \Gamma(X_1, \mathcal{O}_{X_1}^\times))}.$$

This implies that $\ker(\operatorname{Pic}(\operatorname{cosk}_0(X_0)_\bullet) \to \operatorname{Pic}(X_\bullet))$ is finitely generated. In fact, pulling back units from $k^\times$ by γ_1 is injective, as $k \to \Gamma(X_1, \mathcal{O}_{X_1})$ is injective. Thus $\ker(\Gamma(X_0', \mathcal{O}_{X_0'}^\times) \xrightarrow{\gamma_1^*} \Gamma(X_1, \mathcal{O}_{X_1}^\times)) \hookrightarrow \Gamma(X_0', \mathcal{O}_{X_0'}^\times)/k^\times$, which is a finitely generated abelian group. To see this we use the separatedness of X_0 to ensure the existence of a Nagata compactification of X_0', so that we can apply, e.g., [Kah06, Lemme 1]. ■

5.1.11 Proposition. *Let U be a regular, connected k-scheme of finite type, and $\tau : U_\bullet \to U$ a smooth, proper hypercovering such that $\tau_0 : U_0 \to U$ is an alteration (i.e. proper, surjective and* generically finite*) and U_0 is connected. Then the kernel of $\tau^* : \operatorname{Pic} U \to \operatorname{Pic} U_\bullet$ is finitely generated. In particular, $\operatorname{Pic} U_\bullet$ is finitely generated, then so is $\operatorname{Pic} U$.* □

PROOF. If $V \subseteq U$ is the biggest open subset of U such that τ_0 restricted to $V_0 := \tau_0^{-1}(V)$ is flat, then $V \neq \emptyset$, and the complement $U \setminus V$ has codimension ≥ 2. In fact, as $U_0 \to U$ is surjective, for any η mapping to a codimension 1 point $\xi \in U$, the morphism $\mathcal{O}_{U,\xi} \to \mathcal{O}_{U_0,\eta}$ is injective, so $\mathcal{O}_{U_0,\eta}$ is a torsion free $\mathcal{O}_{U,\xi}$-module. But as U is regular, $\mathcal{O}_{U,\xi}$ is a discrete valuation ring, so τ_0 is flat at η, and $\xi \in V$. Thus $\operatorname{Pic}(U) = \operatorname{Pic}(V)$, and $\tau_0|_{V_0}$ is faithfully flat.

Giving an element of $\operatorname{Pic}(\operatorname{cosk}_0(U_0)_\bullet)$ is the same thing as giving an (isomorphism class of) a pair (L, α) with L a line bundle on U_0 and α a descent datum of L relative to U.

Finally, we see that if a line bundle L on U pulls back to the trivial descent datum, then restricting it to V_0 and using faithful flatness shows that $L|_V$ is trivial, so L is trivial, as $U \setminus V$ has codimension ≥ 2. Hence $\operatorname{Pic} U \to \operatorname{Pic}(\operatorname{cosk}_0(U_0)_\bullet)$ is injective, and by Lemma 5.1.10 this implies that the kernel of $\tau^* : \operatorname{Pic} U \to \operatorname{Pic} U_\bullet$ is finitely generated. ■

5.1.12 Proposition. *Let $j : U_\bullet \to X_\bullet$ be a morphism of k-simplicial schemes, such that*

(a) *X_p is regular and proper over k for every p,*

(b) *$j_p : U_p \hookrightarrow X_p$ is an open immersion with dense image,*

(c) *the face maps $X_{i+1} \to X_i$ map $X_{i+1} \setminus U_{i+1}$ to $X_i \setminus U_i$.*

Then the cokernel of the induced map $j^* : \operatorname{Pic} X_\bullet \to \operatorname{Pic} U_\bullet$ is finitely generated. □

PROOF. Let $K_X^i := H^0(X_i, \mathcal{O}_{X_i}^\times)$ and make it into a complex of abelian groups $(K_X, \bar{\delta})$ via $\bar{\delta}_i := \sum_{\ell=0}^{i+1} (-1)^\ell \delta_\ell^* : K_X^i \to K_X^{i+1}$. Define K_U in the analogous fashion (where, to simplify notation, we write δ_i for the faces of $X_\bullet$ and for the faces of $U_\bullet$). Note that the complexes K_X and K_U have finitely generated cohomology groups: For even $i > 0$ we have $k^\times \subseteq \operatorname{im}(\bar{\delta}_{i-1})$, so $H^i(K_X) = \ker(\bar{\delta}_i)/\operatorname{im}(\bar{\delta}_{i-1})$ is a subquotient of $\Gamma(X_i, \mathcal{O}_{X_i}^\times)/k^\times$, which is finitely generated, see e.g. [Kah06, Lemme 1]. The same argument holds for K_U. For odd i, we have $k^\times \cap \ker(\bar{\delta}_i) = 1$, so $\ker(\bar{\delta}_i) \hookrightarrow \Gamma(X_i, \mathcal{O}_{X_i}^\times)/k^\times$, and hence $H^i(K_X)$ and $H^i(K_U)$ are finitely generated as well.

The morphism j induces a morphism of spectral sequences

$$\begin{array}{ccc} E_{1,X}^{p,q} = H^q(X_p, \mathcal{O}_{X_p}^\times) & \Longrightarrow & \mathbb{H}^{p+q}(X_\bullet, \mathcal{O}_{X_\bullet}^\times) \\ \downarrow & & \downarrow \\ E_{1,U}^{p,q} = H^q(U_p, \mathcal{O}_{U_p}^\times) & \Longrightarrow & \mathbb{H}^{p+q}(U_\bullet, \mathcal{O}_{U_\bullet}^\times), \end{array}$$

(note that $K_X = E_{1,X}^{\cdot,0}$, and similarly for K_U) from which we obtain the morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & E_{2,X}^{1,0} = H^1(K_X) \arrow[r] \arrow[d] & \mathrm{Pic}(X_\bullet) \arrow[r] \arrow[d, "j^*"] & \ker(d_{2,X}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & E_{2,U}^{1,0} = H^1(K_U) \arrow[r] & \mathrm{Pic}(U_\bullet) \arrow[r] & \ker(d_{2,U}) \arrow[r] & 0, \end{tikzcd}$$

where $d_{2,X}$ is the differential

$$E_{2,X}^{0,1} = \ker(\operatorname{Pic} X_0 \xrightarrow{\delta_0^* - \delta_1^*} \operatorname{Pic} X_1) \longrightarrow H^2(K_X) = E_{2,X}^{0,2},$$

and similarly for $d_{2,U}$. Since $H^1(K_U)$ is finitely generated, $\operatorname{coker}(H^1(K_X) \to H^1(K_U))$ is also finitely generated, so to finish the proof of the proposition, it remains to show that $\operatorname{coker}(\ker(d_{2,X}) \to \ker(d_{2,U}))$ is finitely generated.

Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(d_{2,X}) \arrow[r] \arrow[d] & \ker(\operatorname{Pic} X_0 \xrightarrow{\delta_0^* - \delta_1^*} \operatorname{Pic} X_1) \arrow[r, "d_{2,X}"] \arrow[d, "\phi_0"'] & \mathrm{im}(d_{2,X}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker(d_{2,U}) \arrow[r] & \ker(\operatorname{Pic} U_0 \xrightarrow{\delta_0^* - \delta_1^*} \operatorname{Pic} U_1) \arrow[r, "d_{2,U}"] & \mathrm{im}(d_{2,U}) \arrow[r] & 0. \end{tikzcd}$$

As $\mathrm{im}(d_{2,X}) \subseteq H^2(K_X)$, we know that $\ker(\mathrm{im}(d_{2,X}) \to \mathrm{im}(d_{2,U}))$ is finitely generated, so by the Snake Lemma, to finish the proof it suffices to show that the middle vertical map $\phi_0 : E_{2,X}^{0,1} \to E_{2,U}^{0,1}$, $\phi_0(L) = L|_{U_0}$ from above has a finitely generated cokernel.

By our regularity assumptions, we have for each i an exact sequence

$$0 \longrightarrow \mathbb{Y}_i \longrightarrow \operatorname{Pic} X_i \longrightarrow \operatorname{Pic} U_i \longrightarrow 0,$$

where $\mathbb{Y}_i$ is the subgroup of $\operatorname{Pic} X_i$ generated by the classes of the (finite number of) codimension 1 points of $X_i \setminus U_i$. In particular, this induces a map

$$\ker(\bar{\delta}_i^* : \operatorname{Pic} U_i \to \operatorname{Pic} U_{i+1}) \longrightarrow \mathbb{Y}_{i+1}/\bar{\delta}_i^* \mathbb{Y}_i,$$

where $\bar{\delta}_i^* = \sum_{\ell=0}^{i+1} (-1)^\ell \delta_\ell^*$. Indeed, we may extend $L \in \ker(\bar{\delta}_i^* : \operatorname{Pic} U_i \to \operatorname{Pic} U_{i+1})$ to some $\tilde{L} \in \operatorname{Pic} X_i$, and map it to $\operatorname{Pic} X_{i+1}$, where it has support contained in $X_{i+1} \setminus U_{i+1}$, i.e. it is mapped to $\mathbb{Y}_{i+1}$. To get a well-defined map on $\operatorname{Pic} U_i$, we have to account for the choice of the extension of L to X_i, that is we have to divide out by the image of $\mathbb{Y}_i$ under $\bar{\delta}_i^*$ which is contained in $\mathbb{Y}_{i+1}$ by assumption (c).

Next, we show that the kernel of this map is precisely the image of the restriction

$$\phi_i : \ker(\operatorname{Pic} X_i \to \operatorname{Pic} X_{i+1}) \longrightarrow \ker(\operatorname{Pic} U_i \to \operatorname{Pic} U_{i+1}).$$

If $L \in \ker(\operatorname{Pic} U_i \to \operatorname{Pic} U_{i+1})$ maps to $\bar{\delta}_i^* M$, for some $M \in \mathbb{Y}_i$, then there is some extension $\tilde{L}$ of L to X_i, such that $\bar{\delta}_i^*(\tilde{L} \otimes M^{-1}) \cong \mathcal{O}_{X_{i+1}}$, so $\tilde{L} \otimes M^{-1} \in \ker(\operatorname{Pic} X_i \to \operatorname{Pic} X_{i+1})$. This shows $L \cong \phi_i(\tilde{L} \otimes M^{-1})$, as M is supported on $X_i \setminus U_i$. Conversely, if some $L \in \ker(\operatorname{Pic} U_i \to \operatorname{Pic} U_{i+1})$ can be extended to $\tilde{L} \in \ker(\operatorname{Pic} X_i \to \operatorname{Pic} X_{i+1})$, then by definition L maps to 0 in $\mathbb{Y}_{i+1}/\bar{\delta}_i{}^* \mathbb{Y}_i$.

This finishes the proof: Specializing the last calculation to $i = 1$, we see that $\operatorname{coker}(\phi_0)$ can be embedded into the finitely generated group $\mathbb{Y}_1/(\delta_0^* - \delta_1^*)\mathbb{Y}_0$. ∎

5.1.2 The Picard group of simply connected varieties

By a simply connected scheme we mean an irreducible scheme X such that $\pi_1^{\text{ét}}(X, \bar{x}) = 1$ for some (or any) geometric point $\bar{x}$ of X. Often we will suppress notation of base points and write π_1 for $\pi_1^{\text{ét}}$. If X is a k-scheme, for some field k, then k is necessarily algebraically closed. We will mostly be interested in the case $\operatorname{char}(k) = p > 0$.

5.1.13 Proposition. *If X is a normal, proper, connected k-scheme of finite type, such that $\pi_1(X)^{ab,(\ell)} = 1$ for some $\ell \neq p$, and $X_\bullet \to X$ a proper hypercovering with X_0 normal, and X_n reduced for all n, then* $\mathrm{NS}(X_\bullet) = \mathrm{Pic}(X_\bullet)$. *In particular,* $\mathrm{Pic}(X_\bullet)$ *is finitely generated.* $\square$

Proof. This is a consequence of cohomological descent: There is an isomorphism $0 = H^1_{\text{ét}}(X, \mu_n) \cong \mathbb{H}^1(X_\bullet, \mu_{n,X_\bullet})$ (see e.g. [BVS01, Lemma 5.1.3]), so $\mathrm{Pic}^0_{X_\bullet/k}(k) = \mathrm{Pic}^{0,\mathrm{red}}_{X_\bullet/k}(k)$ has no ℓ-torsion, and thus is trivial, as $\mathrm{Pic}^{0,\mathrm{red}}_{X_\bullet/k}$ is semi-abelian by Theorem 5.1.8. In fact, if a semi-abelian variety has no ℓ-torsion, then it is an abelian variety, as a nontrivial subtorus would have nontrivial ℓ-torsion. But an abelian variety with trivial ℓ-torsion is trivial. Hence $\mathrm{NS}(X_\bullet) = \mathrm{Pic}(X_\bullet) = \mathrm{Pic}_{X_\bullet/k}(k)$. This group is finitely generated by Proposition 5.1.9. ■

We are ready to prove the main theorem of this section.

Proof (of Theorem 5.1.5.). By Nagata's theorem there exists a proper variety X admitting U as a dense open subscheme, and since U is normal we may assume X to be normal. By [dJ96] there exists an augmented proper hypercovering $X_\bullet \to X$ with X_n regular and proper over k, such that the part Z_n of X_n lying over $X \setminus U$ is a strict normal crossings divisor. Write $U_n := X_n \setminus Z_n$. As U is connected we can pick $X_\bullet$ such that X_0 and U_0 are connected. Also note that $\pi_1(U)^{\mathrm{ab},(\ell)}$ surjects onto $\pi_1(X)^{\mathrm{ab},(\ell)}$ (see, e.g., [SGA1, Prop. V.6.9]), so $\pi_1(X)^{\mathrm{ab},(\ell)} = 1$. We have shown that $\operatorname{Pic} X_\bullet$ is finitely generated (Proposition 5.1.13), that $\operatorname{Pic} X_\bullet$ maps to $\operatorname{Pic} U_\bullet$ with finitely generated cokernel (Proposition 5.1.12) and that this implies that $\operatorname{Pic} U$ is finitely generated (Proposition 5.1.11). ■

Chapter 6

Outlook

To conclude the main narrative of this dissertation, we use the present chapter to collect questions which remain unanswered, and to pose some related questions which extend the content of this text or just appear interesting. For a more comprehensive overview of the questions surrounding the topic of stratified bundles, we refer to the forthcoming article [Esn12], which has some overlap with the discussion in the present chapter.

We divide our exposition into two parts: Foundational problems, which arise from Chapters 3 to 4, and problems of "Gieseker-type" which, roughly, are concerned with the question how strongly the category of regular singular stratified bundles is controlled by its subcategory of finite bundles, i.e. by tamely ramified étale coverings.

6.1 Foundational problems

As always let k be an algebraically closed field of positive characteristic p and X a smooth, separated, finite type k scheme.

6.1.1 Question. *If E is a stratified bundle on X, is it true that E is regular singular if and only if ϕ^*E is regular singular for all morphisms $\phi : C \to X$ with C a regular curve over k?* □

Using results about tame ramification, we provided an affirmative answer to Question 6.1.1 for finite bundles E in Theorem 4.6.5. For general E we showed that E is regular singular if ϕ^*E is regular singular for all ϕ, under the additional assumption that k is uncountable. We can only answer the opposite part of the question under the assumption that X admits a good compactification (i.e. under the assumption that there exists a smooth, proper, finite type k-scheme $\overline{X}$, containing X as a dense open subscheme, such that $\overline{X} \setminus X$ is a strict normal crossings divisor), and it seems plausible that the proof can be extended with weaker techniques for the resolution of singularities, e.g. as in [Tem08].

Since the analogous question in characteristic 0 has a positive answer, and since this answer has an algebraic proof [And07], further investigation of Question 6.1.1 in positive characteristic certainly seems worthwhile.

In a similar vein we can ask:

6.1.2 Question. *If E is a regular singular stratified bundle on X, and* $\mathrm{Exp} \subseteq \mathbb{Z}_p/\mathbb{Z}$ *the subgroup generated by the exponents of E, is it true that* Exp *is a finitely generated group?* □

This question seems reasonable, because if X admits a good compactification $\overline{X}$, the group Exp is generated by the exponents along the irreducible components of $\overline{X} \setminus X$.

Answers to both of these questions would follow easily from a solution to the open problem of resolution of singularities in positive characteristic. For our applications weaker statements than a positive characteristic analogue of Hironaka's theorem could be sufficient; for example we would be amiss not to mention the following question of D. Abramovich and F. Oort, a positive answer of which would be of tremendous help in the study of regular singular stratified bundles:

6.1.3 Question ([AO00, Question 2.9]). *If X is an integral algebraic variety, does there exist an alteration $f : Y \to X$ with Y regular and $k(X) \subseteq k(Y)$ purely inseparable?* □

In light of the results on topological invariance of the categories of stratified bundles and regular singular stratified bundles (e.g. Theorem 3.3.14) a version of Question 6.1.3 which provides control over boundaries (just as A.J. de Jong's original theorem on alteration does) would be almost as good as the knowledge of full resolution of singularities for the purpose of studying regular singular stratified bundles.

6.2 Problems surrounding Gieseker's conjecture

For the sake of this exposition, we systematically ignore base points. A general motivation to a version of Gieseker's conjecture for regular singular stratified bundles has already been given in the introduction to Chapter 5. The main question raised there, which unfortunately remains unanswered, is:

6.2.1 Question (Question 5.0.8). *Is it true that $\pi_1^{\mathrm{tame}}(X) = 0$ if and only if every regular singular stratified bundle on X is trivial?* □

It follows from the results of Chapter 4, that $\pi_1^{\mathrm{tame}}(X) = 0$ if there are no nontrivial regular singular stratified bundles. In fact, for this conclusion to hold, it suffices that there are no nontrivial *finite* regular singular stratified bundles. In Theorem 5.1.1 we proved a partial converse, namely that there are no nontrivial stratified line bundles, if $\pi_1^{\mathrm{tame}}(X, \bar{x}) = 0$. This proof, just like the proof in [EM10] for the projective case, uses in an essential way moduli methods, in the form of the picard scheme. Without a striking new idea to approach the problem, an affirmative answer to Question 6.2.1 seems to require a similar reduction to the projective case, and then the use of moduli methods.

Even though this is out of reach at the moment, it seems justifiable to dream on, and formulate a generalization of Question 6.2.1:

6.2.2 Question. *Let $f : Y \to X$ be a morphism of smooth, separated, finite type k-schemes. Is it true that f induces an isomorphism $f_* : \pi_1^{\mathrm{tame}}(Y) \to \pi_1^{\mathrm{tame}}(X)$ if and only if f induces an equivalence $f^* : \mathrm{Strat}^{\mathrm{rs}}(X) \to \mathrm{Strat}^{\mathrm{rs}}(Y)$?* □

This question specializes to Question 6.2.1, if $Y = \operatorname{Spec}(k)$. If $f^* : \operatorname{Strat}^{\mathrm{rs}}(X) \to \operatorname{Strat}^{\mathrm{rs}}(Y)$ is an equivalence, then again by the results of Chapter 4, f induces an isomorphism on the tame fundamental groups. Note that the results about "topological invariance" proved in Section 3.3.2 give an affirmative answer to Question 6.2.2 if f is an universal homeomorphism.

Moreover, because a positive answer to Question 6.2.1 is known in the rank 1 case, a first approach to the relative version would be a positive answer to the following question:

6.2.3 Question. *Let $f : Y \to X$ be a morphism of smooth, separated, finite type k-schemes, and write* $\operatorname{Pic}^{\mathrm{rs}}(X)$ *for the group of isomorphism classes of stratified line bundles on X. Is it true that f induces an isomorphism of groups*

$$f^* : \operatorname{Pic}^{\mathrm{rs}}(X) \to \operatorname{Pic}^{\mathrm{rs}}(Y)$$

if f induces an isomorphism $f_* : \pi_1^{\mathrm{tame}}(Y) \to \pi_1^{\mathrm{tame}}(X)$? □

Appendix A

Good partial compactifications and discrete valuations

In all of this appendix let k denote an algebraically closed field and X a smooth, separated, finite type k-scheme, if not stated otherwise. Generally, we are interested in normal compactifications of X, i.e. in proper normal k-schemes X^N which contain X as a dense open subscheme, and in the generic points of $X^N \setminus X$. By Nagata's theorem (see e.g. [Lüt93] or [Con07]), such a compactification always exists because X is separated over k. Unfortunately it is not known in general whether there always exists a *smooth* compactification of X. This is known if k has characteristic 0 (Hironaka's theorem), or if X is of dimension ≤ 3. For $\dim X = 1$, it is well-known that there is a unique regular proper model of X. The 2-dimensional case is also classical; we refer for example to [Lip78]. For $\dim X = 3$, the result is fairly recent and the proof is very complicated, see [CP08], [CP09].

Nonetheless, for the sake of studying codimension 1 points "at infinity", we can use "good partial compactifications" which can be obtained from a normal compactification X^N as follows: Let X^N be any normal compactification of X, and let $\eta_1, \ldots, \eta_n$ be the codimension 1 points of X^N not contained in X. Then there are open sets $U_1, \ldots, U_n \subseteq X^N$, such that $\eta_i \in U_j$ if and only if $i = j$, and such that U_i is contained in the smooth locus of X^N. Then $\overline{X} := \bigcup_{i=1}^n U_i \cup X$ is smooth and $\overline{X} \setminus X$ is a strict normal crossings divisor. This will be the model for the definition of a "good partial compactification" of X given below. Of course $\overline{X}$ is not proper, but it enables us to study η_i in a smooth context.

A.1 Definition. We define "good partial compactifications" and make some conventions regarding valuation theory.

- Let $\overline{X}$ be a smooth, separated, finite type k-scheme, and $X \subseteq \overline{X}$ an open subscheme such that the boundary divisor $D := \overline{X} \setminus X$ has strict normal crossings. We denote such a situation by $(X, \overline{X})$ and call it a *good partial compactification.*
- By a *discrete valuation* ν we mean a valuation such that the associated

valuation ring is a discrete valuation ring, in particular one dimensional. This is a discrete valuation of rank 1 in the language of [KS10].

- If X is a connected, normal, separated k-scheme, then we will call a discrete valuation ν on $k(X)$ *geometric*, if there is a normal compactification X^N of X, such that ν corresponds to a codimension 1 point on X^N. We write $\mathrm{DVal}^{\mathrm{geom}}(X)$ for the set of geometric discrete valuations on $k(X)$.
- If $(X, \overline{X})$ is a good *partial* compactification, and η a codimension 1 point in $\overline{X}$, then we write ν_η for the discrete valuation associated with η. If ν is a geometric discrete valuation on $k(X)$ and $\nu = \nu_\eta$, then we say that the good partial compactification $(X, \overline{X})$ *realizes the geometric valuation* η. $\square$

A.2 Proposition ([Liu02, §8, Ex. 3.14]). *If X is a normal connected, separated k-scheme, then a discrete valuation ν on $k(X)$ is geometric if and only if*

$$\mathrm{trdeg}_k\, k(\nu) = \dim X - 1,$$

where $k(\nu)$ is the residue field of ν. $\square$

A.3 Definition. Let X be a normal, separated, finite type k-scheme. We say that two good partial compactifications $(X, \overline{X}_1)$ and $(X, \overline{X}_2)$ are *equivalent* and write $(X, \overline{X}_1) \sim (X, \overline{X}_2)$ if they realize the same geometric discrete valuations of $k(X)$. We write $(X, \overline{X}_1) \leq (X, \overline{X}_2)$ if every geometric discrete valuation realized on $\overline{X}_1$ is also realized on $\overline{X}_2$. Write $\mathrm{PC}(X)$ for the *set of equivalence classes of good partial compactifications of X.* $\square$

Lets see that this actually is well-defined:

A.4 Proposition. *The notion of equivalence on good partial compactifications is an equivalence relation, and "$\leq$" defines a partial order on* $\mathrm{PC}(X)$*. For a finite subset $V \subseteq \mathrm{DVal}^{\mathrm{geom}}(X)$ there is a good partical compactification $(X, \overline{X})$ such that $(X, \overline{X})$ realizes $\nu \in \mathrm{DVal}^{\mathrm{geom}}(X)$, if and only if $\nu \in V$. The resulting map*

$$\theta : (\{\textit{finite subsets of } \mathrm{DVal}^{\mathrm{geom}}(X)\}, \subseteq) \to (\mathrm{PC}(X), \leq)$$

is an isomorphism of partially ordered sets. $\square$

PROOF. It is clear that $\sim$ is an equivalence relation and that $\leq$ is a partial ordering. Also, if there is such a map θ, then it is clearly bijective. Thus the only nontrivial part of the proposition is the statement that for a finite subset $\{\nu_1, \dots, \nu_r\} \subseteq \mathrm{DVal}^{\mathrm{geom}}(X)$ there is a good partial compactification realizing precisely $\nu_1, \dots, \nu_r$. Moreover, given a good partial compactification, realizing $\nu_1, \dots, \nu_r$, we can just remove possible other codimension 1 points to get a good partial compactification realizing precisely $\nu_1, \dots, \nu_r$.

Thus we need to prove that to a finite set $\nu_1, \dots, \nu_r$ there exists a good partial compactification $(X, \overline{X})$ realizing $\nu_1, \dots, \nu_r$. Let X^N be any normal compactifcation of X. Since X^N is proper, there exist points $x_i \in X^N$, such that $\mathcal{O}_{\nu_i} \supseteq \mathcal{O}_{X^N, x_i}$. If all x_i are codimension 1 points, we are done. If not, we want to find suitable blow-ups of X^N to get into this situation.

Since ν_i is geometric, we know that $\mathrm{trdeg}_k\, k(\nu_i) = \dim X - 1$. Since

$$\mathrm{codim}_{X^N} \overline{\{x_i\}} = \dim \mathcal{O}_{X^N, x_i},$$

we see that

$$\dim X - 1 = \operatorname{trdeg}_k(k(\nu_i)) = \operatorname{trdeg}_{k(x_i)}(k(\nu_i)) + \dim X - \dim \mathcal{O}_{X^N,x_i}$$

so

$$\operatorname{trdeg}_{k(x_i)}(k(\nu_i)) = \dim \mathcal{O}_{X^N,x_i} - 1.$$

Then by [Liu02, §8, Ex. 3.14] it follows that there is a finite chain of blow-ups

$$X_m^N \to X_{m-1}^N \to \cdots \to X_1^N \to X^N$$

with centers all lying over $\overline{\{x_i\}}$, such that ν_i is centered in a codimension 1 point of X_m^N. We replace X_m^N by its normalization, and repeat this process for all i, to get a normal compactification $\tilde{X}^N$ of X, such that each ν_i is centered in a codimension 1 point of $\tilde{X}^N$. We then remove a closed subset of codimension ≥ 2 to get the desired good partial compactification. ■

A.5 Corollary. *The partially ordered set* $(\mathrm{PC}(X), \leq)$ *is directed.* □

A.6 Remark. Due to Proposition A.4 we can and will identify the partially ordered set $\mathrm{PC}(X)$ with the set of finite subsets of $\mathrm{DVal}^{\mathrm{geom}}(X)$. □

A.7 Proposition. *Let $f : Y \to X$ be a* dominant *morphism of smooth, separated, finite type k-schemes. For every good partial compactification $(Y, \overline{Y})$ of Y, there exists an equivalent good partial compactification $(Y, \overline{Y}')$, and a good partial compactification $(X, \overline{X})$, such that there is a morphism $\bar{f} : \overline{Y} \to \overline{X}$, with $\bar{f}|_Y = f$.* □

Proof. Fix a normal compactification X^N of X. Then, after replacing $\overline{Y}$ by an open subset with complement of codimension ≥ 2, f extends to a morphism $\bar{f} : \overline{Y} \to X^N$. Let $\eta_1, \ldots, \eta_r$ be the codimension 1 points of $\overline{Y}$ not lying in Y. If $\bar{f}(\eta_i) \in X$, for all i, then we can take $\overline{X} = X$ and we are done. If $\bar{f}(\eta_i)$ is a codimension 1 point for all i with $\bar{f}(\eta_i) \notin X$, we are done: Take $\overline{X}$ be a union of sufficently small neighborhoods of $\bar{f}(\eta_i)$ in X^N containing X. Thus assume that the closure of $x_i := \bar{f}(\eta_i)$ has codimension ≥ 2 in X^N. The discrete geometric valuation ν_{η_i} on $k(Y)$ then induces a discrete valuation on $k(X)$: Since $\bar{f}(\eta_i) \notin X$, $\mathcal{O}_{\overline{Y},\eta_i}$ does not contain $k(X)$, $\mathcal{O}_i := \mathcal{O}_{\overline{Y},\eta_i} \cap k(X)$ is a discrete valuation ring containing $\mathcal{O}_{X^N,x_i}$.

We claim that $\mathcal{O}_i$ satisfies

$$\operatorname{trdeg}_{k(x_i)} k(\mathcal{O}_i) = \dim \mathcal{O}_{X^N,x_i} - 1. \tag{A.1}$$

To prove (A.1) consider the local homomorphism $\mathcal{O}_i \hookrightarrow \mathcal{O}_{\overline{Y},\eta_i}$. We always have the inequality

$$\operatorname{trdeg}_{k(\mathcal{O}_i)} k(\eta_i) \leq \operatorname{trdeg}_k \operatorname{Frac}(\mathcal{O}_{\overline{Y},\eta_i}) - \operatorname{trdeg}_k \operatorname{Frac}(\mathcal{O}_i) = \dim Y - \dim X, \tag{A.2}$$

see [Mat89, Thm. 15.5]. The ring $\mathcal{O}_{\overline{Y},\eta_i}$ is the inductive limit of finitely generated $\mathcal{O}_i$-algebras, so there is some such algebra R and a prime $\mathfrak{p} \in \operatorname{Spec} R$, such that

$$\operatorname{trdeg}_{k(\mathcal{O}_i)} R_{\mathfrak{p}}/\mathfrak{p} = \operatorname{trdeg}_{k(\mathcal{O}_i)} k(\eta_i)$$

and

$$\operatorname{trdeg}_{\operatorname{Frac}(\mathcal{O}_i)} \operatorname{Frac}(R) = \operatorname{trdeg}_{\operatorname{Frac}(\mathcal{O}_i)} \operatorname{Frac}(\mathcal{O}_{\overline{Y},\eta_i}) = \dim Y - \dim X.$$

But then, since R is finitely generated over $\mathcal{O}_i$, and since $\mathcal{O}_i$ is regular and thus in particular universally catenary, [Mat89, Thm. 15.6] implies (A.2) is an equality for $\mathcal{O}_i \hookrightarrow R$, i.e. that

$$\operatorname{trdeg}_{k(\mathcal{O}_i)} k(\eta_i) = \operatorname{trdeg}_{k(\mathcal{O}_i)} R_{\mathfrak{p}}/\mathfrak{p} = \dim Y - \dim X.$$

Now (A.1) follows by looking at trancendence degrees in the tower

$$\begin{tikzcd}
k(\eta_i) \arrow[d, dash, "\dim Y - \dim X"] \\
k(\mathcal{O}_i) \arrow[d, dash] \\
k(x_i) \arrow[d, dash, "\dim X - \dim \mathcal{O}_{X^N,x_i}"] \\
k \arrow[uuu, dash, bend left=60, "\dim Y - 1"]
\end{tikzcd}$$

Since we know that (A.1) holds, we can argue as in the proof of Proposition A.4: We apply [Liu02, §8, Ex. 3.14] to see that there is a finite chain of blow-ups

$$X^N_m \to X^N_{m-1} \to \ldots \to X^N_1 \to X^N$$

with centers all lying above $\overline{\{x_i\}}$, such that $\mathcal{O}_i$ corresponds to a codimension 1 point of X^N_m, and then of the normalization of X^N_m. Replace X^N_m by its normalization, and repeat this process for every i, for which $\bar{f}(\eta_i)$ has codimension ≥ 2.

We obtain a normal compactification $\tilde{X}^N$ of X, such that (after perhaps removing a codimension ≥ 2 closed subset from $\overline{Y}$) f extends to $\bar{f} : \overline{Y} \to \tilde{X}^N$, and $\bar{f}(\eta_i)$ is a codimension 1 point because $\overline{\{\bar{f}(\eta_i)\}} \neq X$ by the assumption that $\bar{f}(\eta_i) \notin X$. Let $\overline{X}$ be an open subset of $\tilde{X}^N$, such that $\overline{X}$ contains X and all codimension 1 points of $\tilde{X}^N$, and such that $(X, \overline{X})$ is a good partial compactification. Then the closed set $Z := \bar{f}^{-1}(\tilde{X}^N \setminus X)$ has codimension ≥ 2 in $\overline{Y}$, so $(Y, \overline{Y}) \sim (Y, \overline{Y} \setminus Z)$, and $\bar{f}$ restricts to a map $\bar{f}' : \overline{Y} \setminus Z \to \overline{X}$, such that $\bar{f}'|_Y = f$. ■

A.8 Remark. We can reformulate the results of Proposition A.7 and its proof: Let $f : Y \to X$ be a dominant morphism of smooth, separated, finite type k-schemes. Define

$$\operatorname{DVal}^{\text{geom}}(Y)_f := \{\nu \in \operatorname{DVal}^{\text{geom}}(Y) | \mathcal{O}_\nu \not\supseteq k(X)\}.$$

For every $\nu \in \operatorname{DVal}^{\text{geom}}(Y)_f$, denote by $\nu|_X$ the discrete valuation on $k(X)$ induced by ν. Then the map $\nu \mapsto \nu|_X$ induces a map of partially ordered sets

$$\begin{aligned}
\operatorname{PC}(f) : \operatorname{PC}(Y) &\to \operatorname{PC}(X) \\
\{\nu_1, \ldots, \nu_r\} &\mapsto \{\nu_i|_X \,|\nu_i \in \operatorname{DVal}^{\text{geom}}(Y)_f\}
\end{aligned}$$

such that for every equivalence class $D \in \mathrm{PC}(X)$ there exist a good partial compactification $(X, \overline{X})$ in $\mathrm{PC}(f)(D)$, a good partial compactification $(Y, \overline{Y}) \in D$, and a morphism $\bar{f} : \overline{Y} \to \overline{X}$ such that $\bar{f}|_Y = f$. $\square$

A.9 Remark. The rather ad hoc discussion of good partial compactifications should perhaps be replaced by more conceptual arguments using the Riemann-Zariski space ([ZS75, Ch.VI, §17]), once one has good notions of regular singularity in nonsmooth points. $\square$

Appendix B

Tame ramification in algebraic geometry

In this appendix we collect the definitions and main properties of tamely ramified coverings of schemes, which are used in the main text. Then main reference is [KS10]. In all of this appendix let k be a perfect field, and all schemes are separated, finite type k-schemes.

B.1 Tameness with respect to a fixed good partial compactification

B.1.1 Definition. Let R be a discrete valuation ring.

(a) If $\varphi : R \to S$ is a finite extension of discrete valuation rings, then φ is called *tame* or *tamely ramified*, if the associated extensions of fraction fields and residue fields are separable, and if the ramification index is prime to the characteristic of the residue field of R.

(b) If $\varphi : R \to S$ is a finite ring extension, then S is a Dedekind domain, and φ is called *tame with respect to* R or *tame with respect to the discrete valuation defined by* R, if for every prime ideal $\mathfrak{p}$ of S lying over the maximal ideal R, the extension of discrete valuation rings $R \to S_{\mathfrak{p}}$ is tame.

(c) Let κ be the residue field of R, $K := \operatorname{Frac} R$ and L a finite separable extension of K, then the integral closure S of R in L is a Dedekind domain, and L/K is called *tame with respect to* R or *tame with respect to the discrete valuation defined by* R, if $R \to S$ is tame.

(d) If $(X, \overline{X})$ is a good partial compactification (see Definition A.1), $f : Y \to X$ a finite étale morphism, and η a generic point of $\overline{X} \setminus X$, then f is called *tamely ramified over* η, if $k(Y)/k(X)$ is tamely ramified with respect to the discrete valuation ring $\mathcal{O}_{\overline{X},\eta} \subseteq k(X)$.

The covering f is called *tamely ramified with respect to* $\overline{X} \setminus X$ (or *tame* for short) if f is tamely ramified over all generic points of $\overline{X} \setminus X$.

We write $\operatorname{Cov}((X, \overline{X}))$ for the category of étale coverings of X which are tamely ramified with respect to $\overline{X} \setminus X$.

B.1.2 Theorem ([GM71, Thm. 2.4.2]). *Let $(X, \overline{X})$ be a good partial compactification. Then (after the choice of a fiber functor)* $\mathrm{Cov}((X, \overline{X}))$ *is a galois category in the sense of [SGA1], and it is naturally a subcategory of the category* $\mathrm{Cov}(X)$ *of all étale coverings of X.* □

B.1.3 Definition. Let $(X, \overline{X})$ be a good partial compactification and $\bar{x}$ be a geometric point of X. Then $\bar{x}$ induces a fiber functor $\mathrm{Cov}((X, \overline{X})) \to \mathbf{FinSet}$, and the associated profinite group is denoted by $\pi_1((X, \overline{X}), \bar{x})$ or by $\pi_1^D(X, \bar{x})$, if $D = \overline{X} \setminus X$. □

B.1.4 Remark. In the context of logarithmic geometry, there is a vast generalization of the notion of coverings which are tamely ramified with respect to a normal crossings divisor. We refer e.g. to [Sti02] for an exposition. □

B.1.5 Proposition. *If $(X, \overline{X}_1)$ and $(X, \overline{X}_2)$ are good partial compactifications of X, and $(X, \overline{X}_1) \sim (X, \overline{X}_2)$ (Definition A.3), then*

$$\mathrm{Cov}((X, \overline{X}_1)) = \mathrm{Cov}((X, \overline{X}_2)$$

as subcategories of $\mathrm{Cov}(X)$. □

B.1.6 Definition. If X is smooth, then for $D \in \mathrm{PC}(X)$ it is well-defined (by Proposition B.1.5) to write $\mathrm{Cov}^D(X)$ for $\mathrm{Cov}((X, \overline{X}))$, if $(X, \overline{X})$ is any good partial compactification of X representing D. Similarly, we write $\pi_1^D(X, \bar{x})$ for $\pi_1((X, \overline{X}), \bar{x})$. If $D = \emptyset$, then $\pi_1^{\emptyset}(X, \bar{x}) = \pi_1(X, \bar{x})$. □

B.1.7 Proposition. *If $D_1, D_2 \in \mathrm{PC}(X)$ and $D_1 \leq D_2$, then $\mathrm{Cov}^{D_2}(X) \subseteq \mathrm{Cov}^{D_1}(X)$ as subcategories of $\mathrm{Cov}(X)$, and there are canonical continuous quotient maps*

$$\pi_1(X, \bar{x}) \twoheadrightarrow \pi_1^{D_1}(X, \bar{x}) \twoheadrightarrow \pi_1^{D_2}(X, \bar{x}).$$

These morphisms make $(\pi_1^D(X, \bar{x}))_{D \in \mathrm{PC}(X)}$ into an inductive system over the directed set $\mathrm{PC}(X)$ in the category of profinite groups. □

Proof. This follows from [SGA1, Exp. V, Prop. 6.9], since the inclusion functors $\mathrm{Cov}^{D_2}(X) \subseteq \mathrm{Cov}^{D_1}(X) \subseteq \mathrm{Cov}(X)$ map connected objects to connected objects. ■

B.1.8 Proposition ([KS10, Prop. 4.2]). *Let $(X, \overline{X})$ be a good partial compactification. For a closed curve $\overline{C} \subseteq \overline{X}$ write $C := \overline{C} \cap X$ and $\tilde{C}$ for the normalization of C.*

Then a finite étale covering $Y \to X$ is tamely ramified with respect to $\overline{X} \setminus X$ if and only if for each closed curve $\overline{C} \subseteq \overline{X}$ not contained in $\overline{X} \setminus X$, the base change $Y \times_X \tilde{C} \to \tilde{C}$ is tamely ramified along the normal crossings divisor $(\overline{X} \setminus X) \times \tilde{C}$. □

We prove the following simple lemma about the local structure of tamely ramified morphisms.

B.1.9 Lemma. *If $(X, \overline{X})$ is a good partial compactification with* irreducible *boundary divisor $D := \overline{X} \setminus X$, and $f : Y \to X$ a morphism tamely ramified with respect to D. Then there exist:*

- *an open set* $\overline{U} \subseteq \overline{X}$, *with* $\overline{U} \cap D \neq \emptyset$. *We write* $U := \overline{U} \cap X$ *and* $V := f^{-1}(U)$.
- *a good partial compactification* $(V, \overline{V})$, *such that the restriction of* f *to* V *extends to a finite morphism* $\bar{f} : \overline{V} \to \overline{U}$, *tamely ramified over* $D \cap \overline{U}$,
- *a good partial compactification* $(W, \overline{W})$ *and a finite morphism* $\bar{g} : \overline{W} \to \overline{V}$, *such that the following diagram commutes*

(B.1)

$$\begin{tikzcd}
\overline{W} \arrow[r, "\bar{g}"] & \overline{V} \arrow[r, "\bar{f}"] & \overline{U} \\
W \arrow[u, hook] \arrow[r, "\bar{g}|_W"] \arrow[rr, bend right, "\substack{\textit{tame, finite}\\ \textit{galois étale}}"'] & V \arrow[u, hook] \arrow[r, "f|_V"] & U \arrow[u, hook]
\end{tikzcd}$$

and such that $f|_V \circ \bar{g}|_W$ *is* galois *étale and tamely ramified with respect to* $\overline{U} \cap D$. □

Proof. Let $\eta \in \overline{X}$ be the generic point of the smooth boundary divisor D. Then $R := \mathcal{O}_{\overline{X},\eta}$ is a discrete valuation ring with field of fractions $k(X)$. Let $k(X)^{tame}$ be the composite (in some fixed algebraic closure of $k(X)$) of all finite extensions L of $k(X)$ which are tame with respect to R and unramified with respect to the discrete valuations coming from codimension 1 points on X. Since f is tamely ramified with respect to $\overline{X} \setminus X$, $k(Y) \subseteq k(X)^{\mathrm{tame}}$, and $k(Y)$ corresponds to an open subgroup $H \subseteq \mathrm{Gal}(k(X)^{\mathrm{tame}}/K)$. Let N be the intersection of all conjugates of H in $\mathrm{Gal}(k(X)^{\mathrm{tame}}/k(X))$. This is a finite intersection, so N is a normal open subgroup of $\mathrm{Gal}(k(X)^{\mathrm{tame}}/K)$ and hence corresponds to a finite extension L of $k(Y)$, such that $L/k(X)$ is *galois*, tame with respect to R, and unramified with respect to the codimension 1 points of X.

Now let $\overline{U} \subseteq \overline{X}$ be an affine open neighborhood of η, say $\overline{U} = \operatorname{Spec} A$, such that $\overline{U} \cap X = \operatorname{Spec} A[x^{-1}]$, for some local defining equation x of D. Then $A \subseteq k(X)$. Since f is finite, $f^{-1}(\overline{U} \cap X)$ is also affine, say $V := f^{-1}(\overline{U} \cap X) =: \operatorname{Spec} B$. Then B is the integral closure of $A[x^{-1}]$ in $k(Y)$, and let $\overline{B}$ be the integral closure of A in $k(Y)$. Define $\overline{V} := \operatorname{Spec} \overline{B}$. Then f extends to $\bar{f} : \overline{V} \to \overline{U}$ as claimed. By construction $\overline{V}$ is normal, but perhaps not smooth. But we may shrink $\overline{U}$ to an affine neighborhood of η, which has smooth preimage under $\bar{f}$. We may then assume that $\overline{V}$ is smooth, and after further shrinking $\overline{U}$ we may assume that $\bar{f}^{-1}(D)$ is a disjoint union of smooth divisors.

We still write $\overline{U} = \operatorname{Spec} A$, $\overline{U} \cap X = \operatorname{Spec} A[x^{-1}]$, $f^{-1}(\overline{U} \cap X) = \operatorname{Spec} B$ and $\overline{V} = \operatorname{Spec} B'$.

Next, let $\overline{C}$ be the integral closure of A in the extension L from above and C the integral closure of A_x in L. Then C and $\overline{C}$ are also the integral closures of B and $\overline{B}$ in $L/k(Y)$, respectively. Defining $W := \operatorname{Spec} A$, $\overline{W} := \operatorname{Spec} \overline{A}$ gives us the diagram (B.1), but perhaps with $\overline{W}$ not smooth. But again we can shrink $\overline{U}$ by a suitable affine open neighborhood of η, to make $\overline{W}$ smooth. This finishes the proof. ■

B.1.10 Proposition. *If* $(X, \overline{X})$ *is a good partial compactification and* $f : Y \to X$ *finite étale, then* f *is tamely ramified with respect to* $\overline{X} \setminus X$ *if and only if the galois closure* $f' : Y' \to X$ *of* f *is tamely ramified with respect to* $\overline{X} \setminus X$. □

Proof. Clearly, if f' is tamely ramified with respect to $\overline{X} \setminus X$ then so is f.

The converse follows as in the first part of the proof of Lemma B.1.9: Again write $k(X)^{\mathrm{tame}}$ for the composite of all finite separable extension of $k(X)$ (in some fixed algebraic closure) which are tamely ramified with respect to the discrete valuations of the boundary divisor $\overline{X} \setminus X$ and unramified with respect to all discrete valuations corresponding to codimension 1 points on X. Then f corresponds to an open subgroup of $\mathrm{Gal}(k(X)^{\mathrm{tame}}/k(X))$ which contains an open subgroup which is normal in $\mathrm{Gal}(k(X)^{\mathrm{tame}}/k(X))$. Thus, f is dominated by a galois covering $f'' : Y'' \to X$, tamely ramified with respect to $\overline{X} \setminus X$, which then has to dominate f' as well. Thus f' is tamely ramified (and in fact $f' = f''$). ■

B.2 Tameness in general

B.2.1 Definition. Let X be a normal, connected finite type k-scheme. A finite étale morphism $f : Y \to X$ is called *tame* or *tamely ramified*, if for every geometric discrete valuation ν on $k(X)$, the extension $k(X) \subseteq k(Y)$ is tamely ramified with respect to ν in the sense of Definition B.1.1, (c). Note that f is tame if and only if f is tame with respect to any partial compactification $(X, \overline{X})$, in the sense of Definition B.1.1, (d). □

We state the special case of the main result of [KS10] which is used in the main text.

B.2.2 Theorem ([KS10, Thm. 4.4]). *Let k be a perfect field, and $f : Y \to X$ an étale covering of separated, regular, finite type k-schemes. Then the following are equivalent:*

(a) *f is tamely ramified.*

(b) *For every regular curve C over k, i.e. a regular, separated, finite type k-scheme of dimension 1, and any morphism $C \to X$, the base change $f_C : Y \times_X C \to C$ is tamely ramified.*

If X admits a good compactification $\overline{X}$, i.e. a smooth proper scheme $\overline{X}$, such that $X \subseteq \overline{X}$ is an open subscheme with complement a normal crossings divisor, then (a) and (b) are equivalent to

(c) *f is tamely ramified with respect to $\overline{X} \setminus X$.* □

B.2.3 Theorem ([KS10, Sec. 7]). *Let k be a field and X a regular, connected, separated, finite type k-scheme. Then the category $\mathrm{Cov}^{\mathrm{tame}}(X)$ of tamely ramified étale coverings of X is a Galois category in the sense of [SGA1].* □

B.2.4 Definition. With the notations from Theorem B.2.3, if $\bar{x} \to X$ is a geometric point of X and $F_{\bar{x}} : \mathrm{Cov}^{\mathrm{tame}}(X) \to \mathbf{Set}$ the fiber functor $F_{\bar{x}}(f : Y \to X) = Y_{\bar{x}}$, then the associated profinite group $\pi_1^{\mathrm{tame}}(X, \bar{x})$ is called the *tame fundamental group*.

For the next proposition, we use the results of Appendix A.

B.2.5 Proposition. *Let X be smooth and $\bar{x}$ a geometric point. If $D \in \mathrm{PC}(X)$ is an equivalence class of good partial compactifications, then the inclusion of categories $\mathrm{Cov}^D(X) \subseteq \mathrm{Cov}^{\mathrm{tame}}(X)$ induces a quotient map*

$$p_D : \pi_1^D(X, \bar{x}) \twoheadrightarrow \pi_1^{\mathrm{tame}}(X, \bar{x}).$$

If $D_1 \leq D_2 \in \mathrm{PC}(X)$, then these maps fit in a commutative diagram

$$\begin{tikzcd}
\pi_1^{D_1}(X, \bar{x}) \arrow[dd, two heads] \arrow[dr, two heads, "p_{D_1}"] & \\
& \pi_1^{\mathrm{tame}}(X, \bar{x}) \\
\pi_1^{D_2}(X, \bar{x}), \arrow[ur, two heads, "p_{D_2}"'] &
\end{tikzcd}$$

where the vertical map is the quotient map from Proposition B.1.7. This induces an isomorphism

$$\varinjlim_{D \in \mathrm{PC}(X)} \pi_1^D(X, \bar{x}) \xrightarrow{\cong} \pi_1^{\mathrm{tame}}(X, \bar{x}) \tag{B.2}$$

in the category of profinite groups. □

Proof. To show that (B.2) is an isomorphism we have to show that $\pi_1^{\mathrm{tame}}(X, \bar{x})$ satisfies the universal property of an inductive limit of the inductive system $(\pi_1^D(X, \bar{x}))_{D \in \mathrm{PC}(X)}$ (see Proposition B.1.7). In other words, for any profinite group G, we need to show that the map

$$\mathrm{Hom}(\pi_1^{\mathrm{tame}}(X, \bar{x}), G) \to \varprojlim_D \mathrm{Hom}(\pi_1^D(X, \bar{x}), G) \tag{B.3}$$

induced by the surjections p_D is a bijection.

Given a compatible system of continuous morphisms $\phi_D : \pi_1^D(X, \bar{x}) \to G$, the image $H := \mathrm{im}(\phi_D)$ is independent of D, because the transition morphisms in the system $(\pi_1^D(X, \bar{x}))_D$ are surjective, and H is a profinite group, since $\ker(\phi_D)$ is a closed subgroup of $\pi_1^D(X, \bar{x})$ by continuity. Since for all $D_1 \leq D_2$ we have a diagram

$$\begin{tikzcd}
& \pi_1^{D_1}(X, \bar{x}) \arrow[dd, two heads] \arrow[dr, two heads, "p_{D_1}"] \arrow[drr, two heads, "\phi_{D_1}"] & & \\
\pi_1(X, \bar{x}) \arrow[ur, two heads] \arrow[dr, two heads] & & \pi_1^{\mathrm{tame}}(X, \bar{x}) \arrow[r, dotted, two heads] & H \\
& \pi_1^{D_2}(X, \bar{x}) \arrow[ur, two heads, "p_{D_2}"] \arrow[urr, two heads, "\phi_{D_2}"'] & &
\end{tikzcd}$$

it follows that H corresponds to a unique (pointed) finite étale covering of X, which is tamely ramified with respect to *every* $D \in \mathrm{PC}(X)$. In other words, the quotient $\pi_1(X, \bar{x}) \twoheadrightarrow H$ factors uniquely through $\pi_1^{\mathrm{tame}}(X, \bar{x})$. Hence (B.3) is bijective. ■

Appendix C

Tannakian categories

In this appendix we recall the definition of a tannakian category and state the results that are needed in the main text. The references used are [SR72], [DM82] and [Del90]. We will mostly be interested in *neutral* tannakian categories, but we do make use of the general notion of a fiber functor.

C.1 Tannakian categories

In all of what follows let k be an arbitrary field. We warn the reader that there is no unity of notation in the literature: For example a "tensor category" in [DM82] is different from a "tensor category" in [Del90].

C.1.1 Definition ([Del90, 2.1]). Let $\mathcal{T}$ be a category.

(a) $\mathcal{T}$ is called *k-linear* if $\mathrm{Hom}(X, Y)$ is a k-vector space for every pair of objects $X, Y \in \mathcal{T}$, and such that the composition maps $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ are k-bilinear.

(b) A *symmetric monoidal category* (this is called *tensor category* in [DM82]) is a tuple $(\mathcal{T}, \otimes, \phi, \psi, (\mathbb{I}, \lambda, \mu))$ with

- A functor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, $(X, Y) \mapsto X \otimes Y$,
- The *associativity constraint* ϕ, i.e. a natural isomorphism

$$\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z$$

  for X, Y, Z objects of $\mathcal{T}$. Moreover ϕ is required to satisfy the pentagon axiom, see [DM82, (1.0.1)].

- The *commutativity constraint* ψ, i.e. a natural isomorphism

$$\psi_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$

  which is required to be compatible with ϕ, i.e. it is required to satisfy the *hexagon axiom* [DM82, (1.0.2)].

- The *identity object* $(\mathbb{I}, \lambda, \mu)$, i.e. an object $\mathbb{I}$ of $\mathcal{T}$ together with natural isomorphisms *(unity constraints)*

$$\lambda_X : X \otimes \mathbb{I} \xrightarrow{\cong} X$$

and

$$\mu_X : \mathbb{I} \otimes X \xrightarrow{\cong} X.$$

For more details see [SR72, Ch. I], [DM82, Ch. 1] or [ML98, Ch. VII].

(c) A symmetric monoidal category is called *rigid* if for every object X of $\mathcal{T}$ there exists an object $X^\vee$ of $\mathcal{T}$ and morphisms $\mathrm{ev} : X \otimes X^\vee \to \mathbb{I}$, $\delta : \mathbb{I} \to X \otimes X^\vee$, such that the compositions

$$X \xrightarrow{X \otimes \delta} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{ev} \otimes X} X$$

and

$$X^\vee \xrightarrow{\delta \otimes X^\vee} X^\vee \otimes X \otimes X^\vee \xrightarrow{X^\vee \otimes \mathrm{ev}} X^\vee$$

are the identity.

(d) If $\mathcal{T}$ and $\mathcal{T}'$ are two symmetric monoidal categories, then a *$\otimes$-functor* $\omega : \mathcal{T} \to \mathcal{T}'$ is a functor ω together with a natural isomorphism

$$\omega(X \otimes Y) \to \omega(X) \otimes \omega(Y),$$

compatible with the commutativity, associativity and unity constraints.

A *morphism of $\otimes$-functors* $\alpha : \omega \to \omega'$ is a natural transformation $\alpha : \omega \to \omega'$ such that for all objects X, Y of $\mathcal{T}$ we have a commutative diagram

$$\begin{tikzcd}
\omega(X) \otimes \omega(Y) \arrow[r, "\cong"] \arrow[d, "\alpha"'] & \omega(X \otimes Y) \arrow[d, "\alpha"] \\
\omega'(X) \otimes \omega'(Y) \arrow[r, "\cong"] & \omega'(X \otimes Y)
\end{tikzcd}$$

where the horizontal arrows are the isomorphisms belonging to the datum of a tensor functor.

(e) If $\mathcal{T}$ is an abelian, k-linear, symmetric monoidal category, then a *fiber functor on a k-scheme* S is a k-linear, exact $\otimes$-functor $\omega : \mathcal{T} \to \mathrm{QCoh}(S)$.

(f) Following the terminology of [Del90], we say that $\mathcal{T}$ is a *tensor category over* k, if $\mathcal{T}$ is a k-linear, abelian, rigid symmetric monoidal category with $\mathrm{End}(\mathbb{I}) = k$.

(g) If $\mathcal{T}$ is a tensor category over k admitting a fiber functor on a k-scheme $S \neq \emptyset$, then $\mathcal{T}$ is called *tannakian category over* k.

(h) If $\mathcal{T}$ is a tannakian category over k, then $\mathcal{T}$ is called *neutral tannakian*, if $\mathcal{T}$ admits a fiber functor on the k-scheme $\mathrm{Spec}(k)$. □

We list a few basic facts about these definitions.

C.1.2 Proposition ([Del90, 2.5]). *A symmetric monoidal category $\mathcal{T}$ is rigid if and only if*

(a) *For every pair of objects X, Y of $\mathcal{T}$ there exists an* internal Hom-object $\underline{\mathrm{Hom}}(X,Y)$ *in $\mathcal{T}$, i.e. an object $\underline{\mathrm{Hom}}(X,Y)$ of $\mathcal{T}$ representing the functor*

$$\mathcal{C}^{\mathrm{opp}} \to \mathbf{Set}, T \mapsto \mathrm{Hom}(T \otimes X, Y).$$

We write the morphism $\underline{\mathrm{Hom}}(X,Y) \otimes X \to Y$ corresponding to $\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}$ as $\mathrm{ev}_{X,Y}$.

(b) *For X_i, Y_i $i = 1, 2$, objects of $\mathcal{T}$, the morphism*

$$\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2) \to \underline{\mathrm{Hom}}(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

corresponding to

$$(\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2)) \otimes (X_1 \otimes X_2) \xrightarrow{\mathrm{ev}_{X_1,Y_1} \otimes \mathrm{ev}_{X_2,Y_2}} Y_1 \otimes Y_2$$

is an isomorphism.

(c) *Every object of $\mathcal{T}$ is reflexive, i.e. for every object X, writing $X^\vee = \underline{\mathrm{Hom}}(X, \mathbb{I})$, the natural morphism $X \to (X^\vee)^\vee$ corresponding to $\mathrm{ev}_{X,\mathbb{I}} \circ \psi : X \otimes X^\vee \to \mathbb{I}$ is an isomorphism.* □

C.1.3 Proposition. *If $\mathcal{T}$ is a tannakian category over k, S a k-scheme and $\omega : \mathcal{T} \to \mathrm{QCoh}(S)$ a fiber functor, then:*

(a) *If $f : T \to S$ is a morphism of schemes, then ω induces a fiber functor $f^*\omega : \mathcal{T} \to \mathrm{QCoh}(S) \xrightarrow{f^*} \mathrm{QCoh}(T)$. Thus, since $S \neq \emptyset$ by definition of a tannakian category, there always exist an affine scheme S' and an S-valued fiber functor; S' can even be taken as* $\mathrm{Spec}\, k'$ *for a suitable (possibly transcendental) extension field of k.*

(b) ([Del90, 2.7]) *ω factors through the full subcategory* $\mathrm{LF}(S)$ *of* $\mathrm{QCoh}(S)$ *whose objects are locally free $\mathcal{O}_S$-modules of finite rank.*

(c) ([Del90, 2.7]) *For every object of $\mathcal{T}$ we have a canonical isomorphism $\omega(X^\vee) \cong \omega(X)^\vee$.*

(d) ([DM82, Prop. 1.13]) *If $\omega' : \mathcal{T} \to \mathrm{LF}(S)$ is a second fiber functor, then every morphism of $\otimes$-functors $\alpha : \omega \to \omega'$ is an isomorphism of $\otimes$-functors.*

(e) ([Del90, 2.13]) *Every object of $\mathcal{T}$ is of finite length, and for every pair of objects X, Y of $\mathcal{T}$, the k-vector space* $\mathrm{Hom}(X,Y)$ *is finite dimensional.* □

C.1.4 Example. The most basic examples to keep in mind are:

- The category Vectf_k of finite dimensional k-vector spaces together with its tensor product is neutral tannakian, and the identity functor $\mathrm{Vectf}_k \to \mathrm{Vectf}_k$ is a fiber functor.
- The category $\mathrm{Repf}_k\, G$ of representations of an affine k-group scheme G on finite dimensional k-vector spaces together with its tensor product structure is neutral tannakian. The functor $\mathrm{Repf}_k\, G \to \mathrm{Vectf}_k$ which forgets the G-structure is a fiber functor. □

C.1.5 Definition. If $\mathcal{T}$ is a tensor category over k, and $\mathcal{T}' \subseteq \mathcal{T}$ a subcategory, then $\mathcal{T}'$ is called *tensor subcategory* if

- $\mathcal{T}' \subseteq \mathcal{T}$ is strictly full, i.e. $\mathcal{T}$ is a full subcategory of $\mathcal{T}$ and if X, Y are objects of $\mathcal{T}$, such that $X \in \mathcal{T}'$ and $X \cong Y$, then $Y \in \mathcal{T}'$.
- $\mathcal{T}'$ is closed under finite tensor products. In particular it contains $\mathbb{I}$ as the "empty" tensor product.
- If $X \in \mathcal{T}'$ then $X^\vee \in \mathcal{T}'$.
- $\mathcal{T}'$ is an abelian subcategory, e.g. if $f : X \to Y$ is an arrow in $\mathcal{T}'$, then the kernel of f computed in $\mathcal{T}'$ is isomorphic to the kernel of f computed in $\mathcal{T}$, etc..

If $\mathcal{T}$ is tannakian, then a fiber functor for $\mathcal{T}$ induces a fiber functor for $\mathcal{T}'$, and we say that $\mathcal{T}'$ is a *tannakian subcategory*, instead of using the perhaps more accurate term "subtannakian category". □

For the study of monodromy groups, the notion of a $\otimes$-generator is essential:

C.1.6 Definition. Let $\mathcal{T}$ be a tensor category on k. If X is an object of $\mathcal{T}$, then denote by $\langle X \rangle_\otimes$ the *tensor category on k generated by X*, i.e. the strictly full subcategory of $\mathcal{T}$ with objects isomorphic to subquotients of objects of the form $P(X, X^\vee)$ with $P(x,y) \in \mathbb{N}[x,y]$. This is the smallest tensor subcategory of $\mathcal{T}$ containing E.

If $\mathcal{T} = \langle X \rangle_\otimes$, then X is called *$\otimes$-generator of $\mathcal{T}$*.

$\mathcal{T}$ is called *$\otimes$-finitely generated* if there exist $X_1, \ldots, X_n$, such that every object of $\mathcal{T}$ is isomorphic to $P(X_1, X_2^\vee, \ldots, X_n, X_n^\vee)$, for $P(x_i, y_i) \in \mathbb{N}[x_i, y_i | i = 1, \ldots, n]$. Clearly, in this case $\mathcal{T} = \langle \bigoplus_{i=1}^n X_i \rangle_\otimes$, so $\otimes$-finitely generated tannakian categories always admit a $\otimes$-generator. □

C.1.7 Theorem ([Del90, Cor. 6.20]). *If a tannakian category on k has a $\otimes$-generator, then it admits fiber functor on* $\operatorname{Spec} k'$*, for k' a finite extension of k.* □

C.1.8 Definition. Let $\mathcal{T}$ be a tensor category over k and S a k-scheme.

(a) If ω_1, ω_2 are two fiber functors $\mathcal{T} \to \mathrm{QCoh}(S)$, then we define the functor $\underline{\mathrm{Isom}}_S^\otimes(\omega_1, \omega_2) : \mathbf{Sch}/S \to \mathbf{Set}$ by

$$\underline{\mathrm{Isom}}_S^\otimes(\omega_1, \omega_2)(u : T \to S) := \left\{ u^*\omega_1 \xrightarrow{\cong} u^*\omega_2 \text{ isomorphism of } \otimes\text{-functors} \right\}.$$

(b) If ω_1, ω_2 are fiber functors on k-schemes S_1, S_2, then we define

$$\underline{\mathrm{Isom}}_k^\otimes(\omega_1, \omega_2) := \underline{\mathrm{Isom}}_{S_1 \times_k S_2}^\otimes(\mathrm{pr}_1^*\,\omega_1, \mathrm{pr}_2^*\,\omega_2).$$

(c) Finally, if ω is a fiber functor on a k-scheme S, then we define

$$\underline{\mathrm{Aut}}_S^\otimes(\omega) := \underline{\mathrm{Isom}}_S^\otimes(\omega, \omega)$$

and

$$\underline{\mathrm{Aut}}_k^\otimes(\omega) := \underline{\mathrm{Isom}}_k^\otimes(\omega, \omega).$$

Note that if $\omega : \mathcal{T} \to \mathrm{Vectf}_k$ is a fiber functor, then $\underline{\mathrm{Aut}}_k^{\otimes}(\omega)$ is a functor from the category of k-schemes to the category of groups. □

The main theorem about neutral tannakian categories for our purposes is:

C.1.9 Theorem ([DM82, Thm. 2.11]). *If $\mathcal{T}$ is a neutral tannakian category with fiber functor $\omega : \mathcal{T} \to \mathrm{Vectf}_k$, then*

(a) *The functor $\underline{\mathrm{Aut}}_k^{\otimes}(\omega)$ is representable by an affine k-group scheme G.*

(b) *ω induces an equivalence $\mathcal{T} \xrightarrow{\cong} \mathrm{Repf}_k G$* □

C.1.10 Remark. Theorem C.1.9 is only a special case of a theorem about $\underline{\mathrm{Aut}}_k^{\otimes}(\omega)$ for a fiber functor ω on an arbitrary k-scheme $S \neq \emptyset$. In this case $\underline{\mathrm{Aut}}_k^{\otimes}(\omega)$ is not necessarily a group functor, but only a groupoid functor, and in order to state the general version of the main theorem, one has to discuss representations of groupoids. We do not use this result, so we just refer to [Del90, Thm. 1.12]. □

C.1.11 Definition. Let $\mathcal{T}$ be a tannakian category over k. We define $\mathcal{T}_{\mathrm{triv}}$ to be the full tannakian subcategory of $\mathcal{T}$ with objects the trivial objects of $\mathcal{T}$, i.e. every object of $\mathcal{T}_{\mathrm{triv}}$ is isomorphic to $\mathbb{I}^n$ for some n. □

C.1.12 Proposition. *If $\mathcal{T}$ is a tannakian category, then there is a unique fiber functor $\omega_{\mathrm{triv}} : \mathcal{T}_{\mathrm{triv}} \to \mathrm{Vectf}_k$, and ω_{triv} is an equivalence.*

Every object V of $\mathcal{T}$ has a maximal trivial subobject V_{triv}, and the assignment $V \mapsto V_{\mathrm{triv}}$ is a $\otimes$-functor $\mathcal{T} \to \mathcal{T}_{\mathrm{triv}}$.

Similarly: Every object V of $\mathrm{Ind}(\mathcal{T})$ has a maximal trivial subobject E_{triv}, and the assigment $V \mapsto V_{\mathrm{triv}}$ defines a $\otimes$-functor $\mathrm{Ind}(\mathcal{T}) \to \mathrm{Ind}(\mathcal{T}_{\mathrm{triv}})$. □

PROOF. If $V \in \mathcal{T}$, and if T_1 and T_2 are trivial subobjects of V, then the image of the induced map $T_1 \oplus T_2 \to V$ is a trivial subobject of V containing T_1 and T_2. But V has finite length by Proposition C.1.3, so there exists a maximal trivial subobject V_{triv}. Alternatively, the the maximal trivial subobject can be described as $\underline{\mathrm{Hom}}(\mathbb{I}, V)$.

Now if $V \in \mathrm{Ind}(\mathcal{T})$, then we can write V as an inductive limit $\varinjlim_{i\in I} V_i$ of objects V_i in $\mathcal{T}$, and $V_{\mathrm{triv}} := \varinjlim_{i\in I} V_{i,\mathrm{triv}}$ is the maximal trivial subobject of V.

The other statements are clear. ■

C.1.13 Definition. We define the functor $H^0(\mathcal{T}, -) : \mathcal{T} \to \mathrm{Vectf}_k$ to be the composition $E \mapsto E_{\mathrm{triv}} \mapsto \omega_{\mathrm{triv}}(E_{\mathrm{triv}})$. □

C.2 Properties of $\mathrm{Repf}_k G$

The main reference for this section is [DM82, Ch. 2], but we also recall some additional relevant material.

As mentioned in Example C.1.4, if G is an affine k-group scheme, then the category $\mathrm{Repf}_k G$ of representations of G on finite dimensional k-vector spaces is neutral tannkian and the forgetful functor

$$\omega : \mathrm{Repf}_k G \to \mathrm{Vectf}_k$$

is a fiber functor.

There is a canonical morphism of functors $G \to \underline{\mathrm{Aut}}_k^{\otimes}(\omega)$ from the category of k-algebras to the category of groups. From Theorem C.1.9 it follows that this is an isomorphism $G \xrightarrow{\cong} \underline{\mathrm{Aut}}_k^{\otimes}(\omega)$ of *k-group schemes*.

C.2.1 Proposition ([DM82, Prop. 2.20]). *With the notations from above,*

(a) *G is of finite type over k if and only if $\mathrm{Repf}_k\, G$ has a $\otimes$-generator (Definition C.1.6).*

(b) *G is finite over k, if and only if there exists an object $X \in \mathrm{Repf}_k\, G$, such that every object of $\mathrm{Repf}_k\, G$ is isomorphic to a subquotient of X^n for some $n \geq 0$.* □

In Section C.3 it will be crucial to be able to work with infinite dimensional representations of G, and to relate the category $\mathrm{Rep}_k\, G$ of all linear representations of k to the category $\mathrm{Repf}_k\, G$ of finite dimensional linear representations. This is done by the following proposition:

C.2.2 Proposition. *We keep the notations from above.*

(a) ([DM82, Cor. 2.4]) *Every linear representation of G is a directed union of finite dimensional representations. In other words:*

$$\mathrm{Ind}(\mathrm{Repf}_k\, G) = \mathrm{Rep}_k\, G.$$

In the sequel we will identify $\mathrm{Repf}_k\, G$ with a full subcategory of $\mathrm{Rep}_k\, G$.

(b) ([DM82, Cor. 2.7]) *Every affine k-group scheme G is a directed inverse $G = \varprojlim_i G_i$, with G_i finite type, affine k-group schemes and surjective transition maps $G_j \twoheadrightarrow G_i$, $j \geq i$.* □

Let G' be a second affine k-group scheme and $f : G \to G'$ a homomorphism. Then f induces a $\otimes$-functor $\omega^f : \mathrm{Repf}_k\, G' \to \mathrm{Repf}_k\, G$.

We recall how properties of f are reflected in ω^f and vice versa:

C.2.3 Proposition ([DM82, Cor. 2.9, Prop. 2.21]). *With the notation of the preceding paragraph:*

(a) *If $F : \mathrm{Repf}_k\, G' \to \mathrm{Repf}_k\, G$ is a $\otimes$-functor such that $\omega^G F = \omega^{G'}$, where ω^G denotes the forgetful functor $\mathrm{Repf}_k\, G \to \mathrm{Vectf}_k$, then there is a unique homomorphism $f : G \to G'$, such that $F = \omega^f$.*

(b) *f is faithfully flat if and only if ω^f is fully faithful and if ω^f induces an equivalence*

$$\langle X' \rangle_{\otimes} \to \langle \omega_f(X') \rangle_{\otimes}$$

for every object X' of $\mathrm{Repf}_k\, G'$.

(c) *f is a closed immersion if and only if every object of $\mathrm{Repf}_k\, G$ is isomorphic to a subquotient of an object of the form $\omega^f(X')$, for $X' \in \mathrm{Repf}_k\, G'$.* □

We write $G = \mathrm{Spec}\, \mathcal{O}_G$, with $\mathcal{O}_G$ a k-algebra (not necessarily of finite type). The group structure on $\mathcal{O}_G$ induces on $\mathcal{O}_G$ the structure of a coalgebra.

C.2.4 Proposition ([DM82, Prop. 2.2]). *In addition to the notations from the previous paragraph, let V be a k-vector space (not necessarily finite dimensional). There is a canonical one-to-one correspondence between the representations of $G \to \mathrm{GL}(V)$ and the $\mathcal{O}_G$-comodule structures on V.* □

Because of this proposition, we will use the terms "G-representation" and "$\mathcal{O}_G$-comodule" synonymously.

C.2.5 Definition. The comultiplication on the k-bialgebra $\mathcal{O}_G$ is given by the diagonal $\Delta : \mathcal{O}_G \to \mathcal{O}_G \otimes_k \mathcal{O}_G$. This also gives $\mathcal{O}_G$ the structure of an $\mathcal{O}_G$-comodule, which we call the *regular representation* $(\mathcal{O}_G, \Delta)$. This comodule corresponds to G acting on itself from the right, and it is an object of $\mathrm{Rep}_k\, G$. □

In the rest of this section, we study the special properties of the regular representation $(\mathcal{O}_G, \Delta)$.

C.2.6 Proposition. *We identify $\mathrm{Repf}_k\, G$ with a full subcategory of $\mathrm{Rep}_k\, G$.*

(a) *If $\mathbb{I} \in \mathrm{Repf}_k\, G$ denotes the trivial representation of rank 1, then $(\mathcal{O}_G, \Delta)$ is an $\mathbb{I}$-algebra in $\mathrm{Rep}_k\, G$, and the only trivial subobject of $(\mathcal{O}_G, \Delta)$ is $\mathbb{I}$.*

(b) ([Wat79, Lemma 3.5]) *If $V \in \mathrm{Repf}_k\, G$, then there exists an $n \geq 0$, such that $V \subseteq \mathcal{O}_G^n$.*

(c) ([dS07, 2.3.2 (d)]) *If $\dim_k V = n$, then $V \otimes (\mathcal{O}_G, \Delta) \cong (\mathcal{O}_G, \Delta)^n$ in $\mathrm{Rep}_k\, G$. More specifically, writing $\nu_1, \nu_2 : \mathrm{Repf}_k\, G \to \mathrm{Rep}_k\, G$ for the functors*

$$\nu_1(V) = V \otimes_k (\mathcal{O}_G, \Delta),\ \nu_2(V) = V^{\mathrm{triv}} \otimes_k (\mathcal{O}_G, \Delta)$$

where V^{triv} denotes the vector space V with the trivial G-action, there exists a natural isomorphism of functors $\nu_1 \cong \nu_2$.

(d) *The following diagram of functors commutes:*

$$\begin{tikzcd} \mathrm{Repf}_k\, G \arrow[r, "\omega^G"] \arrow[d, "-\otimes(\mathcal{O}_G,\Delta)"'] & \mathrm{Vectf}_k \arrow[d, hook] \\ \mathrm{Rep}_k\, G \arrow[r, "(-)^G"'] & \mathrm{Vect}_k \end{tikzcd}$$

where $V^G := H^0(G, V)$ denotes the k-vector space of G-invariants of V and ω^G the forgetful functor. □

Proof. (a) follows from the fact that G acting on itself from the right does not have any invariants. We do not reproduce the proof of for (c), but just mention that, intuitively, if G is a group and G acts on itself, say from the right via $x \mapsto xg^{-1}$, then the map defined by $\phi(v \otimes g) := g^{-1}v \otimes g$ is an isomorphism of G-modules $\phi : V^{\mathrm{triv}} \otimes G \to V \otimes G$, because

$$\phi(h(v \otimes g)) = hg^{-1}v \otimes gh^{-1} = h(g^{-1}v \otimes g) = h\phi(v \otimes g),$$

and because it is easy to write down the inverse ψ with $\psi(v \otimes g) = gv \otimes g$.

Finally, (d) follows from (c), since $(\mathcal{O}_G)^G = k$. ■

C.2.7 Proposition. *Let G, G_1, G_2 be affine k-group schemes and $p_1 : G \twoheadrightarrow G_1$, $p_2 : G \twoheadrightarrow G_2$ faithfully flat morphisms. We consider* $\operatorname{Repf}_k G_1, \operatorname{Repf}_k G_2$ *as tannakian subcategories of* $\operatorname{Repf}_k G$ *(Proposition C.2.3).*

(a) *The strictly full subcategory* $\operatorname{Repf}_k G_i \subseteq \operatorname{Repf}_k G$ *has as objects precisely those representations $V \in \operatorname{Repf}_k G$, such that $V \otimes (\mathcal{O}_{G_i}, \Delta) \cong (\mathcal{O}_{G_i}, \Delta)^n$ in* $\operatorname{Rep}_k G$ *for some $n \geq 0$, and $i = 1, 2$.*

(b) *The following are equivalent:*

  (i) $\operatorname{Repf}_k G_1 \subseteq \operatorname{Repf}_k G_2$ *as strictly full subcategories of* $\operatorname{Repf}_k G$.

  (ii) *There is a faithfully flat morphism $p' : G_2 \twoheadrightarrow G_1$, such that the faithfully flat map $p_1 : G \twoheadrightarrow G_1$ factors as $p'p_2$.*

  (iii) *For every object $V \in \operatorname{Repf}_k G_1$ the representation $V \otimes (\mathcal{O}_{G_2}, \Delta)$ is isomorphic to $(\mathcal{O}_{G_2}, \Delta)^n$ in* $\operatorname{Rep}_k G$ *for some n.* □

PROOF. For (a), note that by Proposition C.2.6 every object V of $\operatorname{Repf}_k G_i$ has the property that $V \otimes (\mathcal{O}_{G_i}, \Delta) \cong (\mathcal{O}_{G_i}, \Delta)^n$ in $\operatorname{Rep}_k G_i \subseteq \operatorname{Rep}_k G$. Conversely, the canonical map $V \to V \otimes (\mathcal{O}_{G_i}, \Delta)$, $v \mapsto v \otimes 1$ makes V into a subobject of $V \otimes (\mathcal{O}_{G_i}, \Delta)$: There is an inclusion $\mathbb{I} \hookrightarrow (\mathcal{O}_{G_i}, \Delta)$, which remains injective after tensoring with the representation V. This shows that V lies in $\operatorname{Repf}_k G_i$ if $V \otimes (\mathcal{O}_{G_i}, \Delta) = (\mathcal{O}_{G_i}, \Delta)^n$, which finishes the proof of (a).

For (b), first assume (i). Denote the inclusion functors as follows:

$$\operatorname{Repf}_k G_1 \overset{i_1}{\hookrightarrow} \operatorname{Repf}_k G_2 \overset{p_2^*}{\hookrightarrow} \operatorname{Repf}_k G.$$

Then $p_2^* i_1 = p_1^*$, and by Proposition C.2.3, for every object V of $\operatorname{Repf}_k G_1$ the restriction of i_1 to $\langle V \rangle_\otimes$ can be written as a chain of equivalences

$$\langle V \rangle_\otimes \xrightarrow{p_1^*} \langle p_1^* V \rangle_\otimes = \langle p_2^* i_1(V) \rangle_\otimes \xrightarrow{(p_2^*)^{-1}} \langle i_1(V) \rangle_\otimes .$$

But again by Proposition C.2.3, this means that i_1 comes from a faithfully flat morphism $p' : G_2 \twoheadrightarrow G_1$, such that $p_1 = p'p_2$. This shows that (i) implies (ii).

Now assume that (ii) holds. Then the morphism p' induces an inclusion $(\mathcal{O}_{G_1}, \Delta) \subseteq (\mathcal{O}_{G_2}, \Delta)$, so if $V \in \operatorname{Repf}_k G_1$, then

$$V \otimes (\mathcal{O}_{G_2}, \Delta) = V \otimes (\mathcal{O}_{G_1}, \Delta) \otimes_{(\mathcal{O}_{G_1}, \Delta)} (\mathcal{O}_{G_2}, \Delta) \cong (\mathcal{O}_{G_2}, \Delta)^n$$

for some n, since $(\mathcal{O}_{G_2}, \Delta)$ is in fact a $(\mathcal{O}_{G_1}, \Delta)$-algebra in $\operatorname{Rep}_k G$. This shows that (ii) implies (iii).

Finally assume that (iii) holds, i.e. that that for every $V \in \operatorname{Repf}_k G_1$ the representation $V \otimes (\mathcal{O}_{G_2}, \Delta)$ is isomorphic to $(\mathcal{O}_{G_2}, \Delta)^n$, for some $n \geq 0$. By (a), we see that $\operatorname{Repf}_k G_1 \subseteq \operatorname{Repf}_k G_2$. ■

C.3 Torsors

In this category we fix a neutral tannakian category $\mathcal{T}$ over k. We start by recalling the definition of a torsor:

C.3.1 Definition. If G is an affine group scheme over k and S a finite type k-scheme, then a *G-torsor* on S is a scheme T, faithfully flat and affine over S, together with a G-action

$$T \times_S G \to T$$

such that the induced morphism

$$T \times_S G \to T \times_S T$$

is an isomorphism. □

The following result states that there is an intimate relationship between fiber functors with values on a scheme S and torsors on S:

C.3.2 Theorem ([DM82, Thm. 3.2]). *Let S be a k-scheme, $\omega : \mathcal{T} \to \mathrm{Vectf}_k$ a fiber functor and $G := \underline{\mathrm{Aut}}_k^{\otimes}(\omega)$.*

(a) *For any fiber functor η on $\mathcal{T}$ with values on S, the sheaf*

$$\underline{\mathrm{Isom}}_k^{\otimes}(\omega, \eta)$$

is representable by an affine scheme faithfully flat over S, and hence an G-torsor on S.

(b) *The functor $\eta \mapsto \underline{\mathrm{Isom}}_k^{\otimes}(\omega, \eta)$ is an equivalence between the category of fiber functors on $\mathcal{T}$ with values on S and the category of G-torsors over S.* □

C.3.3 Remark. The proof in [DM82] is only stated for affine schemes S, but it extends to general schemes S. For a slightly different proof in the scheme case we refer to the nice write-up in [dS07, Prop. 10], where one should take $\mathcal{T} = \mathrm{Fdiv}(S)$. □

We give some relevant details on the construction going into the proof. The main difficulty to show that the functor $\underline{\mathrm{Isom}}_k^{\otimes}(\omega, \rho)$ is representable by a G-torsor. The most conceptual approach starts by generalizing the notions of algebras, spectra, coalgebras, torsors, etc. for objects of an arbitrary neutral tannakian category, as is done for example in [Del89, §5].

C.3.4 Definition. Let $\mathcal{T}$ be a neutral tannakian category.

- A *unital ring in $\mathcal{T}$* is an $\mathbb{I}$-algebra in $\mathrm{Ind}(\mathcal{T})$.
- The category of *affine schemes in $\mathcal{T}$*, denote bye $\mathrm{Aff}_{\mathcal{T}}$ is by definition the opposite category of the category of unital rings in $\mathrm{Ind}(\mathcal{T})$. The affine scheme in $\mathcal{T}$ associated with a unital ring A is denoted by $\mathrm{Sp}(A)$, *the spectrum of A.*
- For $X, S \in \mathrm{Aff}_{\mathcal{T}}$ we write $X(S) := \mathrm{Hom}(S, X)$.
- An *affine group scheme in $\mathcal{T}$* is an group object in $\mathrm{Aff}_{\mathcal{T}}$.
- If H is an affine group scheme in $\mathcal{T}$, then $P \in \mathrm{Aff}_{\mathcal{T}}$ is called *H-torsor* if $P \neq \mathrm{Sp}(0)$ and if P is equipped with an H-action $\rho : P \times H \to P$ such that
$$(\mathrm{pr}_1, \rho) : P \times H \to H \times H$$
is an isomorphism.

- If $V \in \mathcal{T}$ is any object, then $\mathrm{Sym}(V^\vee)$ is a unital algebra, and we write $\mathbf{V}$ for the associated affine scheme ("schéma vectoriel") in $\mathcal{T}$. $\square$

Having these notions at our disposal, we apply them first to $\mathcal{T} = \mathrm{Repf}_k G$. The group scheme G gives rise to a group scheme $\mathbf{G}$ in $\mathrm{Rep}_k G$ (this is an object of $\mathrm{Aff}_{\mathrm{Repf}_k G}$!) and the right regular representation $(\mathcal{O}_G, \Delta)$ is the "ring of functions" a $\mathbf{G}$-torsor $\mathbf{P} := \mathrm{Sp}((\mathcal{O}_G, \Delta))$ in $\mathrm{Rep}_k G$. Then a statement akin to Proposition C.2.6, (c) shows that the forgetful functor $\mathrm{Repf}_k G \to \mathrm{Vectf}_k$ is naturally isomorphic to the functor

$$V \mapsto \mathbf{V} \mapsto (\mathbf{V} \times \mathbf{P})^{\mathbf{G}}.$$

If $\mathcal{T}$ is an arbitrary neutral tannakian category and $\omega : \mathcal{T} \to \mathrm{Vectf}_k$ a fiber functor, then ω induces an equivalence $\omega' : \mathrm{Ind}(\mathcal{T}) \to \mathrm{Rep}_k G$ for $G = \underline{\mathrm{Aut}}_k^{\otimes}(\omega)$. We can transport the objects constructed above to $\mathcal{T}$ via this equivalence (in fact we can do it canonically, see [DM82, Thm. 3.2]), and we get an affine group scheme $\mathbf{G}$ in $\mathcal{T}$, and a $\mathbf{G}$-torsor $\mathbf{P} \in \mathrm{Ind}(\mathcal{T})$, such that $\omega'(\mathbf{P}) = \mathrm{Sp}(\mathcal{O}_G, \Delta)$ and such that for an object $E \in \mathcal{T}$, $\omega(E) = (\omega'(\mathbf{E} \otimes \mathbf{P}))^{\mathbf{G}}$. We write A_ω for the unital ring in $\mathrm{Ind}(\mathcal{T})$ such that $\mathrm{Sp}(A_{\omega_0}) = \mathbf{P}$.

Finally, if $\rho : \mathcal{T} \to \mathrm{Coh}(X)$ is a fiber functor, then ρ extends to $\rho' : \mathrm{Ind}(\mathcal{T}) \to \mathrm{QCoh}(X)$, and $\rho'(A_\omega)$ is a quasi-coherent $\mathcal{O}_X$-algebra, such that $P_{\omega,\rho} := \mathrm{Spec}\, A_\omega$ is a G_X-torsor.

It remains to see that $P_{\omega,\rho} = \underline{\mathrm{Isom}}_k^{\otimes}(\omega, \rho)$: By Proposition C.2.6 there is an isomorphism of fiber functors $\omega \otimes \rho(A_\omega) \cong \rho \otimes \rho(A_\omega)$, and hence an X-morphism $P_{\omega,\rho}$. Since $P_{\omega,\rho}$ and $\underline{\mathrm{Isom}}_k^{\otimes}(\omega, \rho)$ are both G-torsors, it follows that $P_{\omega,\rho} = \underline{\mathrm{Isom}}_k^{\otimes}(\omega, \rho)$.

Using this construction, we can translate the properties of $(\mathcal{O}_G, \Delta)$ into useful properties of the torsor $\underline{\mathrm{Isom}}_k^{\otimes}(\omega, \rho)$:

C.3.5 Proposition. *Let $\mathcal{T}$ be a neutral tannakian category with fiber functors $\omega : \mathcal{T} \to \mathrm{Vectf}_k$ and $\rho : \mathcal{T} \to \mathrm{Coh}(X)$. Write A_ω for the algebra in $\mathrm{Ind}(\mathcal{T})$ such that $\underline{\mathrm{Isom}}_k^{\otimes}(\omega, \rho) = \mathrm{Spec}(\rho(A_\omega))$. Proposition C.2.6 translates to:*

(a) *For every object $E \in \mathcal{T}$ there exists an n, such that $E \subseteq A_\omega^n$.*

(b) *If $E \in \mathcal{T}$, then $E \otimes A_\omega \cong A_\omega^n$, for some $n \geq 0$.*

(c) *The following diagram of functors commutes:*

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\omega"] \arrow[d, "-\otimes A_{\omega_j}"'] & \mathrm{Vectf}_k \arrow[d, hook] \\
\mathrm{Ind}(\mathcal{T}) \arrow[r, "H^0(\mathrm{Ind}(\mathcal{T}), -)"'] & \mathrm{Vect}_k
\end{tikzcd}$$

Here $H^0(\mathrm{Ind}(\mathcal{T}), V)$ denotes the maximal trivial subobject of V, see Proposition C.1.12. $\square$

Appendix D

Formal differential geometry following Berthelot

In [BO78, Ch. II], P. Berthelot defines a categorical framework, in which it makes sense to discuss n-connections, stratifications, differential operators etc. This framework applies to many different setups: To the “classical” notions of connections and differential operators from [EGA4], to P. Berthelot’s notion of PD-connections and PD-differential operators, and also to the notion of logarithmic connections and differential operators, as shown in Chapter 2. Everything in this appendix is taken from [Ber74, Ch. II], except for the comments and example on logarithmic schemes and tannakian categories.

At a few points we simplify the exposition, by reducing the generality. For example, we do not mention “formal categories” or “pseudo-stratifications”.

D.1 Formal groupoids

D.1.1 Definition ([Ber74, Déf. II.1.13]). Let T be a topos (in a fixed universe). A *formal groupoid X in T* consists of the following data:

- A commutative unital ring A in T.
- A projective system $(\mathcal{P}^n_X)_{n\in\mathbb{N}}$ of commutative, unital rings in T, with transition morphism $u_{m,n} : \mathcal{P}^m_X \to \mathcal{P}^n_X$ for $m \geq n$.
- Families of morphisms of rings
    - $d^n_0, d^n_1 : A \to \mathcal{P}^n_X$ (note that d^n_0 and d^n_1 make $\mathcal{P}^n_X$ into a left- and righ-A-algebra),
    - $\pi^n : \mathcal{P}^n_X \to A$,
    - $\delta^{m,n} : \mathcal{P}^{m+n}_X \to \mathcal{P}^n_X \otimes_A \mathcal{P}^m_X$, where the tensor product is computed via the right- and left-A-structure of $\mathcal{P}^n_X$,
    - $\sigma^n : \mathcal{P}^n_X \to \mathcal{P}^n_X$.

These data are subject to the following conditions :

- The transition morphisms $u_{m,n}$ of $(\mathcal{P}^n)_n$ are surjective.
- The morphisms d_0^n, d_1^n, π^n, $\delta^{m,n}$, σ^n commute with the transition morphisms $u_{m,n}$.
- If $q_0^{m,n}$ is the composition

$$\mathcal{P}_X^{m+n} \xrightarrow{u_{m+n,n}} \mathcal{P}_X^m \xrightarrow{x \mapsto x \otimes 1} \mathcal{P}_X^m \otimes_A \mathcal{P}_X^n, \tag{D.1}$$

and if $q_1^{m,n}$ is the composition

$$\mathcal{P}_X^{m+n} \xrightarrow{u_{m+n,m}} \mathcal{P}_X^n \xrightarrow{x \mapsto 1 \otimes x} \mathcal{P}_X^m \otimes_A \mathcal{P}_X^n, \tag{D.2}$$

then the following relations hold:

  - $\pi^n d_0^n = \pi^n d_1^n = \mathrm{id}_A$
  - $\delta^{m,n} d_0^{m+n} = q_0^{m,n} d_0^{m+n}$
  - $\delta^{m,n} d_1^{m+n} = q_1^{m,n} d_1^{m+1}$
  - $(\pi^m \otimes \mathrm{id}_{\mathcal{P}_A^n})\delta^{m,n} = u_{m+n,n}$
  - $(\mathrm{id}_{\mathcal{P}_X^m} \otimes \pi^n)\delta^{m,n} = u_{m+n,m}$
  - $(\delta^{m,n} \otimes \mathrm{id}_{\mathcal{P}_X^q})\delta^{m+n,q} = (\mathrm{id}_{\mathcal{P}_X^m} \otimes \delta^{n,q})\delta^{m,n+q}$
  - $\sigma^n d_0^n = d_1^n$
  - $\sigma^n d_1^n = d_0^n$
  - $\pi^n \sigma^n = \pi^n$

and the following squares commute:

□

$$\begin{tikzcd} \mathcal{P}_X^n \otimes_A \mathcal{P}_X^n \arrow[r, "\mathrm{id} \otimes \sigma^n"] & \mathcal{P}_X^n \\ \mathcal{P}_X^{2n} \arrow[u, "\delta^{n,n}"] \arrow[r, "\pi^{2n}"] & A \arrow[u, "d_0^n"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}_X^n \otimes_A \mathcal{P}_X^n \arrow[r, "\sigma^n \otimes \mathrm{id}"] & \mathcal{P}_X^n \\ \mathcal{P}_X^{2n} \arrow[u, "\delta^{n,n}"] \arrow[r, "\pi^{2n}"] & A \arrow[u, "d_1^n"'] \end{tikzcd}$$

D.1.2 Example. (a) Let $f : X \to S$ be a morphism of schemes. Put $A = \mathcal{O}_X$. This is a ring in the zariski topos of S. The diagonal morphism $\Delta : X \to X \times_S X$ is an immersion, so there is an open set $U \subseteq X \times_S X$, such that Δ gives a closed immersion $X \hookrightarrow U$. Let I be the ideal of the closed immersion $\Delta : X \hookrightarrow U$, and define Δ_X^n to be the closed subscheme of U defined by I^{n+1}, and

$$\mathcal{P}_X^n := \mathcal{P}_{X/S}^n := \Delta^{-1}\mathcal{O}_{\Delta_X^n}.$$

This is easily seen to be independent of the choice of U. The maps d_0^n, d_1^n are the maps corresponding to the projections $\Delta_X^n \subseteq X \times_S X \to X$, π^n is the morphism corresponding to the closed immersion $X \subseteq \Delta_X^n$, σ^n arises from the automorphism of $X \times_S X$ which switches the components. Finally

$\delta^{n,m}$ is defined as follows: There is a closed immersion $v : X \hookrightarrow \Delta^m_X \times_X \Delta^n_X$, defined by the diagram

$$\begin{tikzcd}
X \arrow[drr, hook, bend left] \arrow[dr, dotted] \arrow[ddr, bend right] & & \\
& \Delta^m_X \times_X \Delta^n_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Delta^n_X \arrow[d, "p^n_1"] \\
& \Delta^m_X \arrow[r, "p^m_0"'] & X,
\end{tikzcd}$$

and if J is the ideal of v, then $J^{n+m+1} = 0$, so by the universal property of infinitesimal neighborhoods, there is a unique map $\Delta^m_X \times_X \Delta^n_X \to \Delta^{m+n}_X$, and $\delta^{m,n}$ is defined to be the corresponding map of sheaves.

We denote this formal groupoid by $\widehat{\mathcal{P}}_{X/S}$.

(b) If $f : (X, M_X) \to (S, M_S)$ is a finite type morphism of fine log-schemes, then we show in Section 2.1.2 that the projective system of logarithmic principal parts gives rise to a formal groupoid. We write $\widehat{\mathcal{P}}_{X/S}(\log)$ for this formal groupoid if no confusion is likely. If (S, M_S) is $\operatorname{Spec} k$ with its trivial log-structure, and M_X the fine log-structure associated to a strict normal crossings divisor D on X, then we write $\widehat{\mathcal{P}_{X/k}}(\log D)$ for the associated formal groupoid. □

D.1.3 Definition. Given two formal groupoids $X = (A, \mathcal{P}^n_X, \ldots)$ and $X' = (A', \mathcal{P}^n_{X'}, \ldots)$, we define a *morphism of formal groupoids* to be a morphism of rings $\phi : A \to A'$, and a sequence of morphisms of rings $\psi^n : \mathcal{P}^n_X \to \mathcal{P}^n_{X'}$ commuting with all the morphisms used to define a formal groupoid. □

D.1.4 Example. Let X be a k-scheme and $D \subseteq X$ a strict normal crossings divisor. Then the identity $\mathcal{O}_X \to \mathcal{O}_X$ and the canonical maps $\mathcal{P}_{X/k} \to \mathcal{P}_{X/k}(\log D)$ (Corollary 2.2.4) induce a morphism of formal groupoids

$$\widehat{\mathcal{P}_{X/k}} \to \widehat{\mathcal{P}_{X/k}}(\log D).$$

□

D.1.5 Definition ([Ber74, II.1.1.8]). If $u : T' \to T$ is a morphism of topoi, and $X = (A, \mathcal{P}^n_X, \ldots)$ a formal groupoid in T, then $u^{-1}X := (u^{-1}A, u^{-1}\mathcal{P}^n_X, \ldots)$ is a formal groupoid in T', because u^{-1} commutes with tensor products. □

D.1.6 Example ([Ber74, Ex. II.1.1.9]). Consider the following commutative diagram of schemes (resp. fine log-schemes):

$$\begin{tikzcd}
X' \arrow[r, "u"] \arrow[d] & X \arrow[d] \\
S' \arrow[r] & S.
\end{tikzcd}$$

This induces a morphism of zariski topoi, and in turn a morphism of formal groupoids

$$u^{-1}\widehat{\mathcal{P}_{X/S}} \to \widehat{\mathcal{P}_{X'/S'}}$$

(resp.

$$u^{-1}\widehat{\mathcal{P}_{X/S}}(\log) \to \widehat{\mathcal{P}_{X'/S'}}(\log))$$

□

D.2 n-connections and stratifications

Let $X = (A, \mathcal{P}_X^n, \ldots)$ be a formal groupoid in a topos T. Let $\mathrm{Ring}(T)$ denote the category of (commutative unital) rings in T. We will stay in great generality and define n-connections and stratifications on objects of a category F_A, where F is a cofibered category over $\mathrm{Ring}(T)$:

D.2.1 Definition ([SGA1, VI.10]). Let F, E be categories and $p : F \to E$ a functor. F is called *cofibered category over* E if the opposite functor $p^{\mathrm{opp}} : F^{\mathrm{opp}} \to E^{\mathrm{opp}}$ makes F^{opp} into a fibered category over E. □

D.2.2 Example. (a) The main example the reader should keep in mind is the cofibered category **Mod** of modules over varying rings. The fiber categories $\mathbf{Mod}_A$ are precisely the categories A-modules.

(b) Another example is less obvious: Let k be a field, and $\mathcal{T}$ a tannakian category over k. Write Fib for the fibered category of fiber functors with values in k-algebras. Then Fib is cofibered over the category of rings in the zariski topos $\mathbf{Sch}/k$. We will later see an example of a stratification on a fiber functor (Example D.2.4,(d)). □

D.2.3 Definition ([Ber74, Déf. II.1.2.1, II.1.3.1]). Let $X = (A, \mathcal{P}_X^n, \ldots)$ be a formal groupoid in T, and F a cofibered category over $\mathrm{Ring}(T)$. If $n \geq 1$, then an *n-connection relative to* X on an object M of F_A is an isomorphism

$$\epsilon_n : \mathcal{P}_X^n \otimes_A M \xrightarrow{\cong} M \otimes_A \mathcal{P}_X^n$$

in $F_{\mathcal{P}_X^n}$ (in particular $\mathcal{P}_X^n$-linear), which induces the identity on M via the base change $\pi^n : \mathcal{P}_X^n \to A$.

A *stratification with respect to* X on an object M of F_A is the datum of an n-connection ϵ_n relative to X on M for every $n \geq 1$, such that for $m \leq n$ the diagrams

$$\begin{tikzcd} \mathcal{P}_X^n \otimes M \arrow[r, "\epsilon_n"] \arrow[d] & M \otimes_A \mathcal{P}_X^n \arrow[d] \\ \mathcal{P}^m \otimes_A M \arrow[r, "\epsilon_m"] & M \otimes_A \mathcal{P}_X^m \end{tikzcd}$$

commute, and such that the following *cocycle condition* holds: Recall the definition of the morphisms $q_i^{m,n} : \mathcal{P}_X^{m+n} \to \mathcal{P}_X^m \otimes_A \mathcal{P}_X^n$ from (D.1) and (D.2). Then we require that

$$(\delta^{k,n-k})^*(\epsilon_n) = (q_0^{k,n-k})^*(\epsilon_n) \circ (q_1^{k,n-k})^*(\epsilon)$$

holds in $F_{\mathcal{P}_X^k \otimes \mathcal{P}_X^{n-k}}$ for every $n, k \in \geq 0$. □

D.2.4 Example. (a) If $f : X \to S$ is a smooth morphism of finite type then a stratification on an $\mathcal{O}_X$-module (i.e. on an object of the $\mathbf{Mod}_{\mathcal{O}_X}$, where $\mathbf{Mod}$ is the cofibered category over the category $\mathrm{Ring}(\mathbf{Sch}/S)$) in the sense of [Gro68] by [Ber74, Prop. II.1.3.3].

(b) Let S be a locally noetherian scheme and X a scheme locally of finite type over S. Then by [SGA1, Thm. 8.3], pullback along the closed immersion $X \hookrightarrow \Delta^n_{X/S}$ induces is an equivalence of categories

$$\mathbf{\acute{E}t}(\Delta^n_{X/S}) \xrightarrow{\cong} \mathbf{\acute{E}t}(X),$$

where $\mathbf{\acute{E}t}(X)$ is the category of étale schemes over X. If $p_1^n, p_2^n : \Delta^n_{X/S} \to X$ are the two projections, then both composites

$$X \hookrightarrow \Delta^n_{X/S} \xrightarrow{p_i^n} X$$

are the identity. Thus, if $Y \to X$ is an étale morphism, then there is a canonical $\Delta^n_{X/S}$-isomorphism $\epsilon_n : (p_1^n)^*Y \xrightarrow{\cong} (p_2^n)^*Y$: ϵ_n corresponds to the identity via the bijection

$$\mathrm{Hom}_{\mathbf{\acute{E}t}(\Delta^n_{X/S})}((p_1^n)^*Y, (p_2^n)^*Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{\acute{E}t}(X)}(Y, Y).$$

Moreover, if $m \geq n$, then pullback along the closed $\Delta^n_{X/S} \hookrightarrow \Delta^m_{X/S}$ gives an equivalence $\mathbf{\acute{E}t}(\Delta^m_{X/S}) \to \mathbf{\acute{E}t}(\Delta^n_{X/S})$, and we get a commutative diagram of bijections

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{\acute{E}t}(\Delta^n_{X/S})}((p_0^n)^*Y, (p_1^n)^*Y) \arrow[dr, "\cong"] & \\
& \mathrm{Hom}_{\mathbf{\acute{E}t}(X)}(Y, Y) \\
\mathrm{Hom}_{\mathbf{\acute{E}t}(\Delta^m_{X/S})}((p_0^m)^*Y, (p_1^m)^*Y), \arrow[uu, "\cong"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

so we see that the vertical arrow maps ϵ_m to ϵ_n.

Finally for the cocycle condition, note that for all n, m we have a closed immersion $v : X \hookrightarrow \Delta^m_{X/S} \times_X \Delta^n_{X/S}$ with nilpotent ideal, defined by the diagram

$$\begin{tikzcd}
X \arrow[dr, dotted, "v"] \arrow[drr, hook, bend left] \arrow[ddr, hook, bend right] & & \\
& \Delta^m_{X/S} \times_{p_0^m, X, p_1^n} \Delta^n_{X/S} \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\mathrm{pr}_1"'] & \Delta^m_{X/S} \arrow[d, "p_0^m"] \\
& \Delta^n_{X/S} \arrow[r, "p_1^n"] & X
\end{tikzcd}$$

which induces the closed immersion $w : \Delta^m_{X/S} \times_X \Delta^n_{X/S} \hookrightarrow \Delta^{m+n}_{X/S}$. We get a commutative diagram of functors

$$\begin{tikzcd} & \mathbf{\acute{E}t}(\Delta^m_{X/S}) \arrow[d, "\mathrm{pr}_0^*"'] \arrow[dr, "\cong"] & \\ \mathbf{\acute{E}t}(\Delta^{m+n}_{X/S}) \arrow[r, "\cong", "w^*"'] & \mathbf{\acute{E}t}(\Delta^m_{X/S} \times_X \Delta^n_{X/S}) \arrow[r, "\cong", "v^*"'] & \mathbf{\acute{E}t}(X) \\ & \mathbf{\acute{E}t}(\Delta^n_{X/S}). \arrow[u, "\mathrm{pr}_1^*"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

It is clear that $(v^*)^{-1}$ maps $\mathrm{id}_Y \in \mathrm{Hom}_{\mathbf{\acute{E}t}(X)}(Y, Y)$ to $\mathrm{pr}_0^*(\epsilon_m)\,\mathrm{pr}_1^*(\epsilon_n) \in \mathrm{Hom}(v^*Y, v^*Y)$, and thus that

$$w^*(\epsilon_{m+n}) = \mathrm{pr}_0(\epsilon_m)\,\mathrm{pr}_1^*(\epsilon_n).$$

The upshot of this discussion is that if we consider Y as an $\mathcal{O}_X$-algebra object of $(\mathbf{Sch}/S)^{\mathrm{opp}}$, then Y carries a stratification. (Note that the morphism w corresponds to $\delta^{m,n}$).

If $f : Y \to X$ happens to be an affine morphism, then $f_*\mathcal{O}_Y$ can be considered as a stratified object of the cofibered category of $\mathcal{O}_X$-algebras in $\mathbf{Mod}_{\mathcal{O}_X}$. But our discussion shows more: The stratification on $f_*\mathcal{O}_Y$ is compatible with the algebra structure of $f_*\mathcal{O}_Y$.

(c) Let (S, N) be a fine, saturated, locally noetherian log-scheme, and let (X, M_X) be a fine, saturated, locally notherian log-scheme over (S, N_S). In this situation the same discussion as in the previous example applies to log-étale morphisms over (X, M_X): The closed immersions $(X, M_X) \hookrightarrow \Delta^n_{(X,M_X)/(S,N)}$ constructed in Section 2.1.2 are *strict* closed immersions, and hence "Kummer universal homeomorphisms" by [Vid01, Rem. 2.2]. Then [Vid01, Thm. 0.1] shows that pullback induces equivalences

$$\mathbf{\acute{E}t}^{\log}(\Delta^n_{(X,M_X)/(S,N)}) \to \mathbf{\acute{E}t}^{\log}((X, M_X)),$$

here $\mathbf{\acute{E}t}^{\log}((X, M_X))$ denotes the category of log-étale morphisms from fine, saturated log-schemes to (X, M_X). This is all we need to complete the argument as above. Thus, log-étale schemes naturally carry logarithmic stratifications.

(d) If k is a field and $\mathcal{T} = \mathrm{Strat}(X/k)$ the category of $\mathcal{O}_X$-coherent modules with stratification in the above sense, then $\mathrm{Strat}(X/k)$ is tannakian, and the forgetful fiber functor $\rho : \mathrm{Strat}(X/k) \to \mathrm{Coh}(X)$ carries a stratification in the sense of Definition D.2.3, see [SR72, IV.1.2.3]. □

A different characterization

In many cases, giving an n-connection is equivalent to giving a datum that is very similar to the datum of a classical connection:

D.2.5 Proposition ([Ber74, Prop. II.1.4.2, Cor. II.1.4.4]). *As before, let $X = (A, \mathcal{P}_X^n, \ldots)$ a formal groupoid in T, and F a cofibered category over the category of rings in T. Then the following statements are true:*

(a) *If the kernel I of $\pi^n : \mathcal{P}_X^n \to A$ is locally nilpotent, then giving an n-connection on an A-module M is equivalent to giving a d_1^n-linear morphism*

$$\theta : M \to M \otimes_A \mathcal{P}_X^n$$

reducing to the identity modulo I.

(b) *If M is an object of F_A, then giving a stratification on M is equivalent to giving a family of d_1^n-linear morphisms*

$$\theta_n : M \to M \otimes_A \mathcal{P}_X^n$$

such that

- *For $m \leq n$, the diagram*

$$\begin{tikzcd} & M \otimes_A \mathcal{P}_X^n \arrow[dd] \\ M \arrow[ur, "\theta_n"] \arrow[dr, "\theta_m"'] & \\ & M \otimes_A \mathcal{P}_X^m \end{tikzcd}$$

  commutes.

- *For every $n \geq 0$ the composition*

$$M \xrightarrow{\theta_n} M \otimes_A \mathcal{P}_X^n \xrightarrow{\mathrm{id}_M \otimes \pi^n} M$$

  is the identity.

- *For every m, n the diagram*

$$\begin{tikzcd} M \otimes_A \mathcal{P}_X^{m+n} \arrow[r, "\mathrm{id}_m \otimes \delta^{m,n}"] & M \otimes_A \mathcal{P}_X^m \otimes_A \mathcal{P}_X^n \\ M \arrow[u, "\theta_{m+n}"] \arrow[r, "\theta_n"] & M \otimes_A \mathcal{P}_X^n \arrow[u, "\theta_m \otimes \mathrm{id}"'] \end{tikzcd}$$

  commutes. □

Categories of stratifications

D.2.6 Definition. Let $X = (A, \mathcal{P}_X^n, \ldots)$ be a formal groupoid in T and F a cofibered category over the category of rings in A. If M, M' are objects of

F_A with n-connections ϵ_n, ϵ'_n (resp. stratifications $(\epsilon_n)_{n\geq 0}$ and $(\epsilon'_n)_{n\geq 0}$), then a morphism $f : M \to M'$ in F_A is called *horizontal morphism* if the square

$$\begin{tikzcd} \mathcal{P}^n_X \otimes_A M \arrow[r, "\epsilon_n"] \arrow[d, "\mathrm{id}\otimes f"'] & M \otimes_A \mathcal{P}^n_X \arrow[d, "f\otimes \mathrm{id}"] \\ \mathcal{P}^n_X \otimes_A M' \arrow[r, "\epsilon'_n"] & M' \otimes_A \mathcal{P}^n_X \end{tikzcd}$$

commutes (resp. commutes for all $n \geq 0$).

We write $\mathrm{Strat}(X, F)$ for the category with objects pairs $(M, (\epsilon_n)_{n\geq 0})$, with $M \in F_A$ and $(\epsilon_n)_{n\geq 0}$ a stratification on M with respect to X, and morphism the horizontal morphisms in F_A. Similarly, let $\mathrm{Conn}^n(X, F)$ be the category with objects pairs (M, ϵ_n), where $M \in F_A$ and ϵ_n an n-connection on X with respect to X, and morphisms the horizontal morphisms of F_A. □

The category of modules with n-connection (resp. stratifications) has a tensor product, and in "smooth" situations it is abelian with internal Hom-objects:

D.2.7 Proposition ([Ber74, §II.1.5]). *Let T be a topos, $X = (A, \mathcal{P}^n_X, \ldots)$ a formal groupoid T. Then the following statements are true:*

(a) *If $M, N \in \mathbf{Mod}_A$ are two A-modules carring two n-connections ϵ_n, ϵ'_n (resp. stratifications $(\epsilon_n)_{n\geq 0}$, $(\epsilon'_n)_{n\geq 0}$), then $M \otimes_A N$ carries a canonical n-connection (resp. stratification) via the isomorphism*

$$\underbrace{(\mathcal{P}^n_X \otimes_A M) \otimes_{\mathcal{P}^n_X} (\mathcal{P}^n_X \otimes N)}_{\cong \mathcal{P}^n_X \otimes_A (M\otimes_A N)} \xrightarrow{\epsilon_n \otimes \epsilon'_n} \underbrace{(M \otimes_A \mathcal{P}^n_X) \otimes_{\mathcal{P}^n_X} (N \otimes_A \mathcal{P}^n_X)}_{\cong (M\otimes_A N)\otimes_A \mathcal{P}^n_X}.$$

(b) *If the morphisms $d^n_0, d^n_1 : A \to \mathcal{P}^n_X$ are flat, (resp. flat for every $n \geq 0$), then* $\mathrm{Conn}^n(X, \mathbf{Mod}_A)$ *(resp.* $\mathrm{Strat}(X, \mathbf{Mod}_A)$*) is an abelian category and the kernels and cokernels in* $\mathrm{Conn}^n(X, \mathbf{Mod}_A)$ *(resp.* $\mathrm{Strat}(X, \mathbf{Mod}_A)$*) can be computed in $\mathbf{Mod}_A$.*

(c) *If the $\mathcal{P}^n_X$ are locally free A-modules via d^n_0 and d^n_1, (resp. locally free A-modules via d^n_0,d^n_1 for all $n \geq 0$), then there are natural isomorphisms*

$$\underline{\mathrm{Hom}}_A(M, N) \otimes_A \mathcal{P}^n_X \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathbb{P}^n_X}(M \otimes_A \mathcal{P}^n_X, N \otimes_A \mathcal{P}^n_X)$$

and

$$\mathcal{P}^n_X \otimes_A \underline{\mathrm{Hom}}_A(M, N) \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathcal{P}^n_X}(\mathcal{P}^n_X \otimes_A M, \mathcal{P}^n_X \otimes N).$$

Thus if $M, N \in$ $\mathrm{Conn}^n(X, \mathbf{Mod}_A)$ *(resp. $M, N \in$* $\mathrm{Strat}(X, \mathbf{Mod}_A)$*), then $\underline{\mathrm{Hom}}_A(M, N)$ carries a canonical n-connection (resp. stratification) with respect to X.* □

D.2.8 Definition ([Ber74, II.1.2.5]). Let T be a topos, F a cofibered category over the category of rings in T, and $X = (A, \mathcal{P}^n_X, \ldots)$, $X' = (A', \mathcal{P}^n_{X'}, \ldots)$ two formal groupoids in the topos T, and $\Phi : X \to X'$ a morphism (Definition D.1.3). Recall that $\Phi = (\phi, \psi^n)$, with $\phi : A \to A'$ and $\psi^n : \mathcal{P}^n_X \to \mathcal{P}^n_{X'}$ homomorphisms of rings, compatible with the morphisms $d^n_0, d^n_1, \pi^n, \delta^{m,n}, \sigma^n$, etc.

Let F be a cofibered category over the category of rings in T. For $M \in F_A$ write $M' := M \otimes_A A'$ for the direct image of M in M' along f (in the sense of cofibered categories). Then there are canonical isomorphisms

$$(\mathcal{P}^n_X \otimes_A M) \otimes_{\mathcal{P}^n_X} \mathcal{P}^n_{X'} \xrightarrow{\cong} \mathcal{P}^n_{X'} \otimes_A M \xrightarrow{\cong} \mathcal{P}^n_{X'} \otimes_{A'} M'$$

and

$$(M \otimes_A \mathcal{P}^n_X) \otimes_{\mathcal{P}^n_X} \mathcal{P}^n_{X'} \xrightarrow{\cong} M \otimes_A \mathcal{P}^n_{X'} \xrightarrow{\cong} M' \otimes_{A'} \mathcal{P}^n_{X'}.$$

Thus, if M, N carry n-connections (resp. stratifications) relative to X, then base change along $\psi^n : \mathcal{P}^n_X \to \mathcal{P}^n_{X'}$ defines an n-connection (resp. stratification) on M', N' with respect to X'. □

D.3 Differential operators

As in [Ber74, §II.2.1], we now restrict to F being the cofibered category **Mod** of modules over commutative rings in the topos T.

D.3.1 Definition ([Ber74, Déf. 2.2.1]). Let $X = (A, \mathcal{P}^n_X, \ldots)$ be a formal groupoid in T, M, N two A-modules.

- A *differential operator of order* $\leq n$ *from* M *to* N is a left-A-linear morphism
$$f : \mathcal{P}^n_X \otimes_A M \to N.$$
Here, as usual, the tensor product $\mathcal{P}^n_X \otimes_A M$ is taken with respect to the right-$\mathcal{O}_X$-structure of $\mathcal{P}^n_X$, and for f to be "left-A-linear", it means that f is A linear, when $\mathcal{P}^n_X \otimes_A M$ is considered as a left-A-module.

- $\mathrm{Diff}^n_X(M, N)$ is defined to be the set of differential operators of order $\leq n$ from M to N with respect to X, and

- $\mathscr{D}\!\mathit{iff}^n_X(M, N) := \mathcal{H}om_A(\mathcal{P}^n_X \otimes_A M, N)$ is defined to be the *sheaf of differential operators of order* $\leq n$ *from* M *to* N. $\mathscr{D}\!\mathit{iff}^n_X(M, N)$ is a $\mathcal{P}^n_X$-module, and thus carries a left- and a right-A-action.

- If $m \geq n$, then the transition morphism $\mathcal{P}^m_X \to \mathcal{P}^n_X$ induces inclusions $\mathrm{Diff}^n_X(M, N) \hookrightarrow \mathrm{Diff}^m_X(M, N)$ and $\mathscr{D}\!\mathit{iff}^n_X(M, N) \hookrightarrow \mathscr{D}\!\mathit{iff}^m_X(M, n)$, and it makes $\mathscr{D}\!\mathit{iff}^n_X(M, N)$ into a sub-$\mathcal{P}^m_X$-module of $\mathscr{D}\!\mathit{iff}^m_X(M, N)$.

- We define $\mathscr{D}\!\mathit{iff}_X(M, N) := \varinjlim_n \mathscr{D}\!\mathit{iff}^n_X(M, N)$.

- The elements $\mathrm{Hom}_A(M, N)$ can be considered as differential operators of order 0: Every $f \in \mathrm{Hom}_A(M, N)$ gives a differential operator
$$\mathcal{P}^0_X \otimes_A M \xrightarrow{\pi^0 \otimes \mathrm{id}} M \xrightarrow{f} N,$$
and since $\pi^0 : \mathcal{P}^0_X \to A$ is surjective, this gives injections $\mathrm{Hom}_A(M, N) \subseteq \mathrm{Diff}^0_X(M, N)$, and $\mathcal{H}om_A(M, N) \subseteq \mathscr{D}\!\mathit{iff}^0_X(M, N)$. □

Differential operators as endomorphisms

Let $f : \mathcal{P}^n_X \otimes_A M \to N$ be a differential operator of order $\leq n$. Recall that $d^n_1 : A \to \mathcal{P}^n_X$ denoted the morphism that gives $\mathcal{P}^n_X$ its right A-structure. Tensoring with M and precomposing gives rise to a morphism

$$f^\flat : M \xrightarrow{d^n_1 \otimes \mathrm{id}_M} \mathcal{P}^n_A \otimes M \xrightarrow{f} N.$$

The map $f^\flat$ is not necessarily A-linear, and the association $f \mapsto f^\flat$ is not injective in general.

D.3.2 Example. (a) Let $f : X \to S$ be a morphism of schemes, and $\widehat{\mathcal{P}}_{X/S} = (\mathcal{O}_X, \mathcal{P}^n_{X/S}, \ldots)$ the associated formal groupoid in the zariski topos of S (see Example D.1.2). Then the map $f \mapsto f^\flat$ gives an injective map ($f^{-1}(\mathcal{O}_S)$-linear)

$$\mathscr{D}\!\mathit{iff}^n_{X/S}(M, N) \hookrightarrow \mathscr{E}\!\mathit{nd}_{f^{-1}(\mathcal{O}_S)}(M, N),$$

since $\mathcal{P}^n_{X/S}$ is generated by the image of d^n_1.

In fact, if $M = N$, then it is well-known ([EGA4, Prop. 16.8.8]) that the image of $\mathscr{D}\!\mathit{iff}^n_{X/S}(M, M)$ in $\mathscr{E}\!\mathit{nd}_{f^{-1}(\mathcal{O}_S)}(M)$ is precisely the subsheaf of endomorphisms $\phi \in \mathscr{E}\!\mathit{nd}_{f^{-1}(\mathcal{O}_S)}(M)$, such that for all sections $a \in A$, the endomorphism $\phi_a(m) := m \mapsto \phi(am) - a\phi(m)$ is in $\mathscr{D}\!\mathit{iff}^{n-1}_{X/S}(M)$.

(b) If k is a field and X a smooth, finite type k-scheme with a strict normal crossings divisor $D \subseteq X$, and M, N two $\mathcal{O}_X$-modules, then we write $\mathscr{D}\!\mathit{iff}^n_D(M, N)$ for the differential operators from M to N in the formal groupoid $(\mathcal{O}_X, \widehat{P}^n_{X/k}(\log D), \ldots)$. Note that $\mathcal{P}^n_{X/k}(\log D)$ is in general not generated by $d^n_1(\mathcal{O}_X)$: In fact, we have an inclusion $\mathcal{P}^n_{X/k} \hookrightarrow \mathcal{P}^n_{X/k}(\log D)$.

If X is smooth, say with global coordinates $x_1, \ldots, x_m$, such that D is cut out by $x_1, \ldots, x_r$, then, writing $\xi_i := 1 \otimes x_i - x_i \otimes 1 \in \mathcal{P}^n_{X/k}$, $\mathcal{P}^n_{X/k}$ is generated (as a left- and as a right-$\mathcal{O}_X$-algebra) by monomials of order $\leq n$ in $\xi_1, \ldots, \xi_m$, and $\mathcal{P}^n_{X/k}(\log D)$ is generated by monomials in $\xi_1/x_1, \ldots, \xi_r/x_r, \xi_{r+1}, \ldots, \xi_m$ of order $\leq n$, see Proposition 2.2.3.

This shows that $\mathcal{P}^n_{X/k}(\log D)$ is not generated by d^n_1, but it also shows that the map $f \mapsto f^\flat$ is an injective map $\mathscr{D}\!\mathit{iff}^n_D(M, N) \to \mathcal{H}om_k(M, N)$: By the local description, it follows that every differential operator

$$f : \mathcal{P}^n_{X/k}(\log D) \otimes_{\mathcal{O}_X} M \to N$$

gives rise to a unique differential operator

$$f' : \mathcal{P}^n_{X/k} \otimes_{\mathcal{O}_X} M \to N,$$

and the map $f \mapsto f^\flat$ factors as $f \mapsto f' \mapsto f^\flat$. □

The relationship between classical differential operators and logarithmic differential operators seen in the previous example is an instance of the following more general fact:

D.3.3 Proposition ([Ber74, II.2.1.4]). *Let $\Phi = (\mathrm{id}_A, \phi^n)$ be a morphism $X \to X'$ of two formal groupoids $X = (A, \mathcal{P}_X^n, \ldots)$, $X' = (A', \mathcal{P}_{X'}^n, \ldots)$ with $A = A'$. Then for A-modules M, N, there is a $\mathcal{P}_X^n$-linear morphism*

$$\mathscr{D}\!\mathit{iff}(\Phi) : \mathscr{D}\!\mathit{iff}_{X'}^n(M,N) \to \mathscr{D}\!\mathit{iff}_X^n(M,N),$$

where $\mathscr{D}\!\mathit{iff}(\Phi)(f)$ is the composition

$$\mathcal{P}_X^n \otimes_A M \xrightarrow{\phi^n \otimes \mathrm{id}_M} \mathcal{P}_{X'}^n \otimes_A M \xrightarrow{f} N. \qquad \square$$

Ring structure

The map $\delta^{m,n} : \mathcal{P}_X^{m+n} \to \mathcal{P}_X^m \otimes_A \mathcal{P}_X^n$ belonging to the datum of a formal groupoid induces a ring structure on $\mathscr{D}\!\mathit{iff}_X(M,M) := \varinjlim_n \mathscr{D}\!\mathit{iff}_X^n(M,M)$:

D.3.4 Proposition ([Ber74, Prop. 2.1.6]). *If f is a differential operator of order $\leq n$ and g a differential operator of order $\leq m$, then the composition*

$$\mathcal{P}_X^{n+m} \otimes_A M \xrightarrow{\delta^{n,m} \otimes \mathrm{id}_M} \mathcal{P}_X^n \otimes_A \mathcal{P}_X^m \otimes_A M \xrightarrow{\mathrm{id}_{\mathcal{P}_X^n} \otimes g} \mathcal{P}_X^n \otimes_A M \xrightarrow{f} M$$

is a differential operator of order $\leq n+m$, and this induces a ring structure on $\mathscr{D}\!\mathit{iff}_X(M,M)$. The $\mathcal{P}_X^n$-structure of $\mathscr{D}\!\mathit{iff}_X^n(M,M)$ induces a left-A and a right-A-algebra structure on $\mathscr{D}\!\mathit{iff}(M,M)$. $\square$

In the previous section we saw how to associate with a differential operator f from M to M as an (not necessarily A-linear) endomorphism $f^\flat$ of M. The product is compatible with this construction:

D.3.5 Proposition (*loc. cit.*). *The map $\mathscr{D}\!\mathit{iff}_X(M,M) \to \mathrm{End}(M)$, $f \mapsto f^\flat$ is an A-algebra morphism (with respect to both A-structures of $\mathscr{D}\!\mathit{iff}_X(M,M)$). In other words:*

$$(f \cdot g)^\flat = f^\flat \circ g^\flat. \qquad \square$$

$\mathscr{D}$-modules and stratifications

Let T continue to denote a topos and $X = (A, \mathcal{P}_X^n, \ldots)$ a formal groupoid in T. From an n-connection ϵ_n on an A-module M, one constructs a canonical morphism

$$\mathscr{D}\!\mathit{iff}_X^n(A,A) \to \mathscr{D}\!\mathit{iff}_X^n(M,M)$$

as follows: If $f \in \mathscr{D}\!\mathit{iff}_X^n(A,A)$, then the composition

$$\mathcal{P}_X^n \otimes_A M \xrightarrow{\epsilon_n} M \otimes_A \mathcal{P}_X^n \xrightarrow{\mathrm{id}_M \otimes f} M$$

is an element of $\mathscr{D}\!\mathit{iff}_X^n(M,M)$, because f is left-A-linear.

D.3.6 Proposition ([Ber74, Cor. II.2.2.2]). *Using the same notations as above, an n-connection ϵ_n on M defines a $\mathcal{P}_X^n$-linear morphism*

$$\nabla_{\epsilon_n} : \mathscr{D}\!\mathit{iff}^n(A,A) \to \mathscr{D}\!\mathit{iff}_X^n(M,M)$$

which assigns to an element $a \in A$, considered as a differential operator of order 0 (Definition D.3.1), the differential operator of order 0 "multiplication with a". $\square$

We have an analogous statement about stratifications:

D.3.7 Proposition ([Ber74, Cor. II.2.2.5]). *If we keep the previous notations, then with a stratification $\epsilon = (\epsilon_n)_{n\geq 0}$ on M there is functorially associated a morphism of left- and righ-$\mathcal{O}_X$-algebras*

$$\nabla_\epsilon : \mathscr{D}\!\mathit{iff}_X(A, A) \to \mathscr{D}\!\mathit{iff}_X(M, M),$$

sending a section $a \in A$ to the differential operator "multiplication by a". □

Finally, we are interested under which conditions a converse to Proposition D.3.7 holds:

D.3.8 Proposition ([Ber74, Prop. II.4.1.2]). *Let X be formal groupoid in T, $X = (A, \mathcal{P}^n_X, \dots)$ and M and A-module. Assume that for $n \in \mathbb{N}$, $\mathcal{P}^n_X$ is locally free of finite rank as an left-A-module. Then the following statements are true:*

(a) *Giving a stratification $\epsilon = (\epsilon_n)_{n\geq 0}$ on M relative to X is equivalent to giving a ring morphism*

$$\nabla_\epsilon : \mathscr{D}\!\mathit{iff}_X(A, A) \to \mathscr{D}\!\mathit{iff}_X(M, M)$$

reducing to a $\mathcal{P}^n_X$-linear morphism

$$\nabla_n : \mathscr{D}\!\mathit{iff}^n_X(A, A) \to \mathscr{D}\!\mathit{iff}^n_X(M, M)$$

for every n.

(b) *If the kernel of $\pi^n : \mathcal{P}^n_X \to A$ is locally nilpotent, then giving an n-connection on M relative to X is equivalent to giving a $\mathcal{P}^n_X$-linear morphism*

$$\nabla_n : \mathscr{D}\!\mathit{iff}^n_X(A, A) \to \mathscr{D}\!\mathit{iff}^n_X(M, M)$$

which maps for every $a \in A$, the differential operator of order 0 associated with a to the map "multiplication by a"-map $M \to M$. □

D.3.9 Remark. If $f : X \to S$ is a morphism of schemes, and $(\mathcal{O}_X, \mathcal{P}^n_{X/S}, \dots)$ the associated formal groupoid, then if M is an A-module, we saw that there is an inlcusion $\mathscr{D}\!\mathit{iff}_{X/S}(M, M) \subseteq \mathscr{E}\!\mathit{nd}_{f^{-1}(\mathcal{O}_S)}(M)$. Given a map of rings

$$\psi : \mathscr{D}_{X/S} = \mathscr{D}\!\mathit{iff}_{X/S}(\mathcal{O}_X, \mathcal{O}_X) \to \mathscr{E}\!\mathit{nd}_{f^{-1}(\mathcal{O}_S)}(M),$$

which is both left-$\mathcal{O}_X$ and righ-$\mathcal{O}_X$-linear (in particular, sends "multiplication by $a \in \mathcal{O}_X$" to "multiplication by $a \in \mathcal{O}_X$", then the image of ψ lies in $\mathscr{D}\!\mathit{iff}_{X/S}(M, M)$. Indeed, if $f \in \psi(\mathscr{D}^1_{X/S})$, then for every section a of $\mathcal{O}_X$, the adjoint f_a given by $f_a(m) = f(am) - af(m)$ lies in the image of $\mathscr{D}^0_{X/S} = \mathcal{O}_X$. Thus, by [EGA4, Prop. 16.8.8], $f \in \mathscr{D}\!\mathit{iff}^1_{X/S}(M)$. Now we can continue inductively.

If k is a field, X a k-scheme, and D is a strict normal crossings divisor on X, then the same discussion applies to maps $\mathscr{D}_{X/k}(\log D) \to \mathscr{E}\!\mathit{nd}_k(M)$ (see Proposition 2.3.2).

It follows that if $X \to S$ is smooth (or just "differentially smooth of finite type", see [EGA4, §16.10]), then in both examples, giving a stratification on M

with respect to $\mathcal{P}^n_{X/S}$ (resp. $\mathcal{P}^n_{X/k}(\log D)$) is equivalent to giving a $\mathscr{D}_{X/S}$-module structure (resp. a $\mathscr{D}_{X/k}(\log D)$-module structure) on M, compatible with the $\mathcal{O}_X$-structure on M. In other words: Giving a stratification on M is equivalent to giving a left- and right-$\mathcal{O}_X$-linear morphism of rings

$$\mathscr{D}_{X/S} \to \mathscr{E}nd_{f^{-1}(\mathcal{O}_S)}(M)$$

resp.

$$\mathscr{D}_{X/k}(\log D) \to \mathscr{E}nd_k(M).$$

□

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